Q.1 (20 points) Barbara and Dianne go target shooting. Each of Barbara’s shots hits the wooden duck target with probability $1/5$ while each shot of Dianne’s hits it with probability $1/3$. They shoot simultaneously and independently at the same target. Let $B$ be the event that Barbara hits the wooden duck, and let $D$ be the event that Dianne hits the wooden duck.

(a) Find $P(B)$ and $P(D)$.

$P(B) = \frac{1}{5}$ and $P(D) = \frac{1}{3}$.

(b) Find $P(B \cap D)$. Justify your answer.

Since $B$ and $D$ are independent, we have

$$P(B \cap D) = P(B)P(D) = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}.$$  

(c) Can you express the event “the wooden duck was hit” in terms of $B$ and $D$? Find the probability that the wooden duck was hit.

$$P(B \cup D) = P(B) + P(D) - P(B \cap D) = \frac{1}{5} + \frac{1}{3} - \frac{1}{15} = \frac{7}{15}.$$  

(d) Given that the wooden duck was hit, what is the conditional probability that Barbara’s shot hit the duck?

$$P(B | B \cup D) = \frac{P(B)}{P(B \cup D)} = \frac{\frac{1}{5}}{\frac{7}{15}} = \frac{3}{7}.$$  

Q.2 (15 points) If a drawn card is numbered $j$ on the $j$-th trial, you win. At each trial the previously drawn card is returned, the deck of 52 cards is thoroughly shuffled, and a new card is drawn. You win the game if at least “one win” occurs during 13 trials.

(a) Let $A_i$ be the event that a card numbered $i$ is observed on the $i$-th attempt. Find $P(A_i)$ and $P(A_i^c)$ for each $i = 1, 2, \ldots, 13$.

$P(A_i) = \frac{4}{52} = \frac{1}{13}$ and $P(A_i^c) = 1 - P(A_i) = \frac{12}{13}$ for each $i = 1, 2, \ldots, 13$.

(b) Let $B$ be the event that you never win in all the 13 trials. Derive a formula for $P(B)$.

Observe that

$$B = A_1^c \cap A_2^c \cap \cdots \cap A_{13}^c.$$  

Since they are independent, we have

$$P(A_1^c \cap A_2^c \cap \cdots \cap A_{13}^c) = P(A_1^c) \times P(A_2^c) \times \cdots \times P(A_{13}^c) = \left(\frac{12}{13}\right)^{13}.$$  

Thus, we obtain $P(B) = \left(\frac{12}{13}\right)^{13}$.

(c) Derive a formula for the probability that you win the game.

$$P(B^c) = 1 - P(B) = 1 - \left(\frac{12}{13}\right)^{13}.$$  

Q.3 (15 points) Let $X$ and $Y$ be random variables. The joint density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} 
x + y & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1; \\
0 & \text{otherwise}. \\
\end{cases}$$  

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(a) Find \( P(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) \).
\[
P(X \leq \frac{1}{2}, Y \leq \frac{1}{2}) = \int_0^{1/2} \int_0^{1/2} (x + y) \, dx \, dy = \frac{1}{8}
\]

(b) Find the marginal density function for \( X \).
\[
f_X(x) = \int_0^1 (x + y) \, dy = x + \frac{1}{2}.
\]

(c) Find \( E[X] \).
\[
E[X] = \int_0^1 x \left( x + \frac{1}{2} \right) \, dy = \frac{7}{12}
\]

Q.4 (20 points) Let \( X \) and \( Y \) be two random variables. We know that \( E[X] = \frac{1}{2}, E[X^2] = \frac{7}{4}, E[Y] = 1, E[Y^2] = 2, \) and \( E[XY] = \frac{3}{4} \).

(a) Find \( E[X + Y] \).
\[
E[X + Y] = E[X] + E[Y] = \frac{3}{2}
\]

(b) Find \( \text{Var}(X) \).
\[
\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{3}{2}
\]

(c) Find \( \text{Cov}(X, Y) \).
\[
\text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{1}{4}
\]

(d) Find \( \text{Var}(X + Y) \).
\[
\text{Var}(Y) = E[Y^2] - (E[Y])^2 = 1. \text{ Thus, } \text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y) = 3.
\]

Q.5 (15 points) Two boys play basketball in the following way. They take turns shooting and stop when a basket is made. Adam goes first and has probability \( p_1 \) of making a basket on any round. Bob goes, who shoots second, has probability \( p_2 \) of making a basket. The outcomes of the successive trials are assumed to be independent.

(a) What is the probability that a winner is determined by the first round? That is, find the probability that Adam wins at his first attempt or Bob wins at his first attempt. Express it in terms of \( p_1 \) and \( p_2 \).

Let \( A \) be the event that Adam wins at his first attempt, and let \( B \) be the event that Bob wins at his first attempt. Then \( A \cap B = \emptyset \), and
\[
P(A \cup B) = P(A) + P(B) = p_1 + (1 - p_1)p_2 = p_1 + p_2 - p_1p_2
\]

(b) Given that a winner is determined by the first round, what is the conditional probability that Adam wins? Express it in terms of \( p_1 \) and \( p_2 \). Justify your answer.
\[
P(A | A \cup B) = \frac{P(A)}{P(A \cup B)} = \frac{p_1}{p_1 + p_2 - p_1p_2}
\]
(c) Let \( X \) be the number of rounds to determine the winner. What is the distribution for \( X \)? Can you find the frequency function \( p(i) = P(X = i) \)?

The chance of being determined in one round is the success probability, which is calculated in (a). Thus, \( X \) has a geometric distribution with the frequency function

\[
p(i) = (1 - p_1 - p_2 + p_1 p_2)^{i-1} (p_1 + p_2 - p_1 p_2)
\]

**Q.6** (15 points) Professor Rice was told that he has only 1 chance in 1,000 of being trapped in a much-maligned elevator in any particular day. Assume that the outcomes are mutually independent. If he use the elevator 200 days a year.

(a) What is the distribution for the number of misfortunes for Professor Rice being trapped in a year?

The number \( X \) of misfortunes can follow a binomial distribution with \( n = 200 \) and \( p = 1/1000 \). When \( n = 200 \) is large and \( p = 1/1000 \) is small, a Poisson distribution with \( \lambda = (1/1000)(200) = 0.2 \) is as good as the exact binomial distribution.

(b) What is the probability that he will never be trapped in a year? Justify your answer, and express it in terms of \( e^M \).

\[
P(X = 0) = e^{-0.2}
\]

(c) What is the probability that he is trapped at least once in 10 years? Justify your answer, and express it by using the form \( e^N \).

Let \( Y \) be the number of misfortunes for Professor Rice being trapped in 10 years. Then \( Y \) has a Poisson distribution with \( \lambda = (1/1000)(200)(10) = 2 \). Thus, we obtain

\[
P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-2}
\]