Q.1 (6 points) Let \( A = \begin{bmatrix} 25 & -9 \\ -9 & 9 \end{bmatrix} \), and let \( Q(x, y) = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} \) be a quadratic form.

(a) In the form of \( Q(x, y) = ax^2 + 2bxy + cy^2 \), find \( a \), \( b \), and \( c \).

\[ a = \frac{5}{12}, \quad b = -\frac{1}{4}, \quad c = \frac{1}{4} \]

(b) Calculate \( \begin{bmatrix} 4 \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix} \). What do we call this product of two matrices?

This is the Cholesky decomposition of \( \begin{bmatrix} 25 & -9 \\ -9 & 9 \end{bmatrix} \).

(c) In the form of \( Q(x, y) = (gx)^2 + (hx + ky)^2 \), find \( g \), \( h \), and \( k \).

By using the result of (b) we have

\[
Q(x, y) = \frac{1}{(12)^2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{(12)^2} \begin{bmatrix} 4x & (-3x + 3y) \\ (-3x + 3y) & 4x \end{bmatrix}
\]

\[
= \frac{1}{(12)^2} [(4x)^2 + (-3x + 3y)^2] = \left( \frac{x}{3} \right)^2 + \left( \frac{-x + y}{4} \right)^2
\]

Thus, we find \( g = 1/3, \quad h = -1/4, \quad \text{and} \quad k = 1/4 \)

Q.2 (4 points; 2 extra points) Let \( Q(x, y) \) be the quadratic form of Q.1, and let \( f(x, y) = \frac{1}{24\pi} \exp \left( -\frac{1}{2} Q(x, y) \right) \) be a joint density for \((X, Y)\).

Applying the result of Q.1(c) we can immediately see that

\[
f(x, y) = \frac{1}{24\pi} \exp \left( -\frac{x^2}{2(3)^2} \right) \times \exp \left( -\frac{(y-x)^2}{2(4)^2} \right)
\]

must correspond to \( f_X(x) \times f_{Y|X}(y|x) \).

(a) The marginal density \( f_X(x) \) has a normal density. Find \( \mu_x \) and \( \sigma_x^2 \).

By looking at the exponent we can compare \( \frac{x^2}{2(3)^2} = (x - \mu_x)^2 / 2\sigma_x^2 \); thus, we find \( \mu_x = 0 \), and \( \sigma_x^2 = (3)^2 \)

(b) The conditional density \( f_{Y|X}(y|x) \) has a normal density. Find \( \mu \) (in terms of \( x \)) and \( \sigma^2 \).

Again by looking at the exponent we can compare \( \frac{(y-x)^2}{2(4)^2} = (y-\mu)^2 / 2\sigma^2 \); thus, we find \( \mu = x \) and \( \sigma^2 = (4)^2 \)

(c) (2 extra points) Can you guess \( \rho = \text{Cov}(X, Y) / \sqrt{\text{Var}(X)\text{Var}(Y)} \)?

If you recall that \( f_{Y|X}(y|x) \) has a normal density with mean \( \mu = \mu_y + \rho \sigma_x \sigma_y (x - \mu_x) \) and variance \( \sigma_y^2 = \sigma_x^2 (1 - \rho^2) \), then you can recognize \( \rho \sigma_x \sigma_y = 1 \) and \( \sigma_y \sqrt{1 - \rho^2} = 4 \) from the result of (b). Therefore, we obtain \( \rho \sigma_x \sigma_y = 1 \) and \( \rho = \frac{3}{4} \), and solve it to find \( \rho = 3/5 \).