Q.1  (5 points) Let $X$ be a random variable. We know that $E[X] = \frac{1}{4}$ and $E[X^2] = \frac{1}{2}$.

(a) Find $\text{Var}(X)$.

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{7}{16}.$$  

(b) Define $\text{Cov}(X, Y)$ in terms of expectation.


(c) Let $Y = 2X + 1$. Then find $\text{Cov}(X, Y)$.

$$E[Y] = E[2X + 1] = 2E[X] + 1 = \frac{3}{2} \quad \text{and} \quad E[XY] = E[2X^2 + X] = 2E[X^2] + E[X] = \frac{5}{4}. \quad \text{Thus,} \quad \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = \frac{7}{8}.$$  

Or, we can apply the formula

$$\text{Cov}(a_1X + b_1, a_2Y + b_2) = a_1a_2\text{Cov}(X, Y),$$

and immediately obtain $\text{Cov}(X, 2X + 1) = 2\text{Cov}(X, X) = 2\text{Var}(X) = \frac{7}{8}.$

Q.2  (5 points) Let $X$ and $Y$ be random variables. The joint density function of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} \frac{3x^2 + xy}{20} & \text{if } 0 \leq x \leq 2 \text{ and } 0 \leq y \leq 2; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $P(X \leq 2, Y \leq 2)$.

$$P(X \leq 1, Y \leq 1) = \int_0^2 \left\{ \int_0^2 \frac{3x^2 + xy}{20} \, dx \right\} \, dy = \int_0^2 \frac{y + 4}{10} \, dy = \left[ \frac{y^2}{20} + \frac{4y}{10} \right]_{y=0}^{y=2} = 1.$$  

Or, since $X$ and $Y$ take values between 0 and 2, we must have $P(X \leq 2, Y \leq 2) = 1.$

(b) Find the marginal density $f_Y(y)$ of $Y$.

$$f_Y(y) = \int_0^2 \frac{3x^2 + xy}{20} \, dx = \left[ \frac{x^3}{20} + \frac{x^2y}{40} \right]_{x=0}^{x=2} = \frac{y + 4}{10} \quad \text{if } 0 \leq y \leq 2; \quad \text{otherwise, } f_Y(y) = 0.$$  

(c) Find $E[Y]$.

$$\int_0^2 y f_Y(y) \, dy = \int_0^2 \frac{y^2 + 4y}{10} \, dy = \left[ \frac{y^3}{30} + \frac{y^2}{5} \right]_{y=0}^{y=2} = \frac{16}{15}.$$