2^k Factorial Designs

Introduction:

- Suppose we have a number of factors controlling a process (say, factors A, B, C, D, ...).

- Example - A Chemical Process
  Factor A - temperature
  Factor B - concentration of one or more chemical constituents
  Factor C - catalysts
  Factor D - reaction times
  etc.

- We will assume that each factor is crossed with all of the other factors.

- Each level of each factor appears in all other levels of every other factor.

- Example

<table>
<thead>
<tr>
<th>Concentration</th>
<th>Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>30°C</td>
</tr>
<tr>
<td>50 cc/l</td>
<td>y_{11}</td>
</tr>
<tr>
<td>100 cc/l</td>
<td>y_{21}</td>
</tr>
<tr>
<td>150 cc/l</td>
<td>y_{31}</td>
</tr>
</tbody>
</table>

- We can have factorial arrangements of treatments in designs such as CRD, RCBD, LS, and C-0.

- Let \( a \) be the number of levels of Factor A, \( b \) be the number of levels of Factor B, and so on... . Some of the potential hypotheses we may be interested are

  1. Main Effects:
     \[ H_0 : \mu_{A_1} = \mu_{A_2} = \ldots = \mu_{A_a} \quad H_0 : \mu_{B_1} = \mu_{B_2} = \ldots = \mu_{B_b} \]

  2. Two-Way Interactions:
     \[ H_0 : \text{Factor A does not interact with Factor B} \]

  3. Three-Way Interactions: (if there are three treatments)

Problems incurred with these types of experiments.

- Suppose we have \( a \) levels of A, \( b \) levels of B, etc.

- The total number of treatments = \( a \times b \times c \times \ldots \).

- If you have a large number of factors, the total can be very large.

- In a \( 2^k \) Factorial Design, we will limit the number of levels of each factor to 2.
• That is, we have 2 levels per treatment.
• These designs are typically used to “screen” potential factors.
• In other words, which factors are important?
• The advantages are that we can minimize the number of treatments and the computations of sums of squares are relatively easy.

Calculating the ANOVA Table.

The entire ANOVA table for a $2^k$ Factorial design can be calculated using contrasts.

Recall that a treatment contrast is calculated as $\psi = \sum c_i \mu_i$ where $\mu_i$ are the treatment means and $\sum c_i = 0$.

Notation:

• Let $A_1$ be the first level of Factor A, $A_2$ be the second level of Factor A.
• Similarly, $B_1$ and $B_2$ are the first and second levels of Factor B, respectively.
• Since the factors are crossed, then if we have $k = 2$, we have $2^2 = 4$ treatment combinations given by $A_1B_1$, $A_1B_2$, $A_2B_1$, and $A_2B_2$.
• For constructing contrasts for the ANOVA, many texts switch to an alternative notation.
• Suppose a given treatment combination receives a small letter corresponding to each factor appearing at its high level. Then for $k = 2$ we have

<table>
<thead>
<tr>
<th>Notation</th>
<th>$A_1B_1$</th>
<th>$A_1B_2$</th>
<th>$A_2B_1$</th>
<th>$A_2B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alternative Notation</td>
<td>(1)</td>
<td>$b$</td>
<td>$a$</td>
<td>$ab$</td>
</tr>
</tbody>
</table>

Contrast Construction:

Using the notation above (for $a$, $b$, above - and possibly $c$ if we have $k = 3$, etc), let $X$ be an arbitrary effect. That is, $X$ may be equal to Factor $A$, or possibly the interaction term $AB$.

The contrast for $X$ is given by

$$\psi_X = (a \pm 1)(b \pm 1)(c \pm 1)\ldots$$

where the sign, “±” in each parenthesis will be

(−) if the corresponding factor appears in $X$, and

(+) if the corresponding factor appears otherwise.

Example - $2^3$ Factorial design

Suppose we have three factors $A$, $B$, and $C$ each at two levels. Consider the following:
Using our formula above we have

1. $\psi_A = (a - 1)(b + 1)(c + 1)$.
2. $\psi_B = (a + 1)(b - 1)(c + 1)$.
3. $\psi_C = (a + 1)(b + 1)(c - 1)$.
4. $\psi_{AB} = (a - 1)(b - 1)(c + 1)$.
5. $\psi_{AC} = (a - 1)(b + 1)(c - 1)$.
6. $\psi_{BC} = (a + 1)(b - 1)(c - 1)$.
7. $\psi_{ABC} = (a - 1)(b - 1)(c - 1)$.

The sums of squares for a particular contrast is given by

$$SS_\psi = \frac{r \hat{\psi}^2}{2^k}$$

where $r$ is the number of replicates for the mean used in the contrast.

Example - Machine Lifetimes
An engineer is interested in the effect of (A) cutting speed, (B) tool geometry, and (C) cutting angle on the lifetime of a machine tool. A completely randomized design was carried out with a $2^3$ factorial arrangements of treatments. The following data were collected.
A few calculations:
Let \( r \) denote the number of replicates per treatment combination. Let \( t \) be the total number of treatments which in this case \( t = 2^3 = 8 \).

- The grand sum \( y_{...} = 980 \)
- Total sum of squares \( \sum_{i=1}^{8} \sum_{j=1}^{3} y_{ij}^2 - y_{...}^2 / tr = 2095.33 \)
- Sum of squares due to treatments

\[
SS_{\text{treat}} = \frac{1}{r} \sum_{i=1}^{t} y_{i.}^2 - y_{...}^2 / tr = 1612.667
\]

- Sum of squares due to error

\[
SS_{\text{error}} = \sum_{i=1}^{t} \sum_{j=1}^{r} y_{ij}^2 - \frac{1}{r} \sum_{i=1}^{t} y_{i.}^2 = 482.667
\]

The treatment sums, means, and related contrast coefficients are summarized below.

<table>
<thead>
<tr>
<th>contrasts</th>
<th>( A_1B_1C_1 )</th>
<th>( A_1B_1C_2 )</th>
<th>( A_1B_2C_1 )</th>
<th>( A_1B_2C_2 )</th>
<th>( A_2B_1C_1 )</th>
<th>( A_2B_1C_2 )</th>
<th>( A_2B_2C_1 )</th>
<th>( A_2B_2C_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>sum</td>
<td>78</td>
<td>127</td>
<td>119</td>
<td>164</td>
<td>104</td>
<td>113</td>
<td>148</td>
<td>127</td>
</tr>
<tr>
<td>mean</td>
<td>26.000</td>
<td>42.333</td>
<td>39.667</td>
<td>54.667</td>
<td>34.667</td>
<td>37.667</td>
<td>49.333</td>
<td>42.333</td>
</tr>
</tbody>
</table>

Calculation of Sums of Squares

<table>
<thead>
<tr>
<th>Source</th>
<th>Formula</th>
<th>Details</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( SS_A )</td>
<td>( \frac{3}{2r} \psi_A^2 )</td>
<td>( 3(1.333)^2/8 )</td>
<td>0.667</td>
</tr>
<tr>
<td>( SS_B )</td>
<td>( \frac{3}{2r} \psi_B^2 )</td>
<td>( 3(45.333)^2/8 )</td>
<td>770.667</td>
</tr>
<tr>
<td>( SS_C )</td>
<td>( \frac{3}{2r} \psi_C^2 )</td>
<td>( 3(27.333)^2/8 )</td>
<td>280.167</td>
</tr>
<tr>
<td>( SS_{AB} )</td>
<td>( \frac{3}{2r} \psi_{AB}^2 )</td>
<td>( 3(-6.667)^2/8 )</td>
<td>16.667</td>
</tr>
<tr>
<td>( SS_{AC} )</td>
<td>( \frac{3}{2r} \psi_{AC}^2 )</td>
<td>( 3(-35.333)^2/8 )</td>
<td>468.167</td>
</tr>
<tr>
<td>( SS_{BC} )</td>
<td>( \frac{3}{2r} \psi_{BC}^2 )</td>
<td>( 3(-10.333)^2/8 )</td>
<td>48.167</td>
</tr>
<tr>
<td>( SS_{ABC} )</td>
<td>( \frac{3}{2r} \psi_{ABC}^2 )</td>
<td>( 3(-8.667)^2/8 )</td>
<td>28.167</td>
</tr>
</tbody>
</table>
ANOVA

<table>
<thead>
<tr>
<th>Source</th>
<th>d.f.</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>.667</td>
<td>.667</td>
<td>.02</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>770.667</td>
<td>770.667</td>
<td>25.55</td>
<td>**</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>280.167</td>
<td>280.167</td>
<td>9.29</td>
<td>**</td>
</tr>
<tr>
<td>AB</td>
<td>1</td>
<td>16.667</td>
<td>16.667</td>
<td>.55</td>
<td></td>
</tr>
<tr>
<td>AC</td>
<td>1</td>
<td>468.167</td>
<td>468.167</td>
<td>15.52</td>
<td>**</td>
</tr>
<tr>
<td>BC</td>
<td>1</td>
<td>48.167</td>
<td>48.167</td>
<td>1.60</td>
<td></td>
</tr>
<tr>
<td>ABC</td>
<td>1</td>
<td>28.167</td>
<td>28.167</td>
<td>.93</td>
<td></td>
</tr>
<tr>
<td>Error</td>
<td>16</td>
<td>482.667</td>
<td>30.167</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>23</td>
<td>2095.333</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Conclusions:
1. Factors B and C (tool geometry and cutting angle) are important factors.
2. Factor B is the most important (examine the magnitudes of the sums of squares).
3. There is significant interaction between Factor A (cutting speed) and C (cutting angle).
4. A plot of the A by C means reveals the following:
   at a low cutting angle, increasing the speed yields an increase in lifetime.
   at a high cutting angle, increasing speed yields decreases in lifetime.
What should you recommend?