Theorem 7. Every open ball is an open set. Every closed ball is a closed set, where a closed ball is defined to be a set of the form \( \{ p : p \) is a point and \( d(x,p) \leq \varepsilon \} \).

Theorem 8. Let \( A \) and \( B \) be closed sets. Then \( A \cap B \) and \( A \cup B \) are closed sets.

Theorem 9. Let \( A \) and \( B \) be open sets. Then \( A \cap B \) and \( A \cup B \) are open sets.

1.5 Notations for some sets. We will adopt the following notation for this class (most of which are quite standard). The notation \( x \in A \) means that the point \( x \) is a member, or element, of the set \( A \). The symbol \( \mathbb{Z} \) denotes the set of all integers (\( \mathbb{Z} = \{ 0,1,-1,2,-2,\ldots \} \)), \( \mathbb{N} \) denotes the strictly positive integers (\( \mathbb{N} = \{ 1,2,3,\ldots \} \)) and \( \omega \) denotes the non-negative integers (\( \omega = \mathbb{N} \cup \{ 0 \} \)). Also, \( \mathbb{R} \) denotes the set of all real numbers (we will say more later about what the term “real number” really means), \( \mathbb{Q} \) denotes the set of all rational numbers (\( \mathbb{Q} = \{ x \in \mathbb{R} : x = n/m \) for some \( n,m \in \mathbb{Z} \} \)), and \( \mathbb{P} \) denotes the set of all irrational numbers (\( \mathbb{P} = \{ x \in \mathbb{R} : x \notin \mathbb{Q} \} \)).

1.6 Definition. Let \( A_1, A_2, A_3, \ldots \) be an infinite sequence of sets. Then the infinite union and the infinite intersection of this sequence are defined by:

\[
\bigcup_{n=1}^{\infty} A_n \overset{\text{def}}{=} \{ x : x \in A_n \text{ for some } n \in \mathbb{N} \} \quad \bigcap_{n=1}^{\infty} A_n \overset{\text{def}}{=} \{ x : x \in A_n \text{ for every } n \in \mathbb{N} \}
\]

1.7 Definition. A set \( A \) is bounded means that there is some point \( x \) and some number \( r > 0 \) such that \( A \subset B_r(x) \).

Proposition 10. Let \( A_1, A_2, \ldots \) be an infinite sequence of closed sets. Then: (a) \( \bigcup_{n=1}^{\infty} A_n \) is a closed set, and (b) \( \bigcap_{n=1}^{\infty} A_n \) is a closed set.

Proposition 11. Let \( A_1, A_2, \ldots \) be an infinite sequence of open sets. Then: (a) \( \bigcup_{n=1}^{\infty} A_n \) is an open set, and (b) \( \bigcap_{n=1}^{\infty} A_n \) is an open set.

Proposition 12. Let \( A_1, A_2, \ldots \) be an infinite sequence of non-empty closed and bounded sets. Then \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \). Furthermore, the condition that the sets are bounded is necessary.

Proposition 13 (Bruggink’s proposition). Let \( A \) and \( B \) be sets with \( A \subset B \). Then \( L(A) \subset L(B) \).