AFFECT, BEHAVIORAL SCHEMAS,

AND

THE PROVING PROCESS

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In this largely theoretical paper, we discuss the relation between a kind of affect, behavioral schemas, and aspects of the proving process. We begin with affect as described in the mathematics education literature, but soon narrow our focus to a particular kind of affect – nonemotional cognitive feelings. We then mention the position of feelings in consciousness because that bears on the kind of data about feelings that students can be expected to be able to express. Next we introduce the idea of *behavioral schemas* as enduring mental structures that link situations to actions, in short, habits of mind, that appear to drive many mental actions in the proving process. This leads to a discussion of the way feelings can both help cause mental actions and also arise from them. Then we briefly describe a design experiment – a course intended to help advanced undergraduate and beginning graduate students to improve their proving abilities. Finally, drawing on data from the course, along with several interviews, we illustrate how these perspectives on affect and on behavioral schemas appear to explain, and are consistent with, our students' actions.

Keywords: tertiary level, design experiment, proof, proving, affect, nonemotional cognitive feelings, behavioral schemas, habits of mind.

1. Introduction

The proving process plays a significant role in both learning and teaching many tertiary mathematical topics, such as abstract algebra or real analysis. In studying such topics, advanced undergraduate and beginning graduate mathematics students not only read and check proofs, but also supplement those proofs with their own subproofs. Their teachers not only explain proofs, but also ask them to construct proofs, especially as a way of assessing the students' understandings. As a consequence, we are designing a course to help advanced undergraduate and beginning graduate students improve their proving abilities We are also exploring the proving process, attending to whatever contributes to that process, in particular, affect and student actions. In this paper, we focus on a particular kind of affect – nonemotional cognitive feelings – and on the implementation of actions via behavioral schemas. We begin with perspectives on affect from the literature.

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2. Perspectives on affect

2.1 Beliefs, attitudes, emotions, and values

Affect is often seen as separate from, but related to, cognition. McLeod [1] treated affect as having three main aspects: (1) *beliefs*, for example, the belief that mathematics is based on rules or that teaching is telling; (2) *attitudes*, for example, dislike of geometric proving or enjoyment of problem solving; and (3) *emotions*, for example, the joy of, or frustration with, solving nonroutine problems. He described beliefs, attitudes, and emotions as of increasing intensity and decreasing stability, with emotions being the most intense and changing the most rapidly.

There are many interesting connections between affect, in the above view, and the teaching, learning, and doing of mathematics. For example, while teaching a college algebra course, the first author occasionally gave group quizzes on which students were to solve moderately nonroutine problems. As she walked around answering questions and facilitating the group work, she found that one student was not contributing to her group's solutions, but was rather very quietly repeating to herself, 'I hate this stuff. I hate this stuff.' This student was expressing a negative emotion, and her attention was clearly divided between expressing the emotion and working on the problem at hand. This, no doubt, overburdened her working memory and interfered with trying to solve the problem, and hence, contributing to the group's efforts.

In the above example, it is very likely that the details of the algebra problem and how it was being attempted were, more or less, immaterial to the student. Thus, affect, in this case an emotion, was associated in a rather 'large-grained', and negative, way with the doing of mathematics, that is, with attempting to solve almost all algebra problems. Much of the research on the relationship between affect and the doing of mathematics seems to have taken a similar large-grained approach [1–5]. However, in this paper we take a finer-grained approach, and we begin an analysis of the relationship between specific kinds of affect and parts of the proving process.

Here is an example that seems to call for taking a finer-grained view of the proving process, and also for a somewhat expanded perspective on affect. During tutoring by the third author, a student, Sofia, from our Spring 2008 course on proving, had produced what we call the *formal-rhetorical* part of a proof, that is, the part of a proof that depends only on unpacking and using the logical structure of the statement of a theorem and associated definitions. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld [6, p. 74]. We call the remaining part of a proof the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved [7].

After writing the formal-rhetorical part of the proof, Sofia needed an idea to continue with the problem-centered part of the proof. We inferred that Sofia was aware of needing an idea because she had had experience writing the formal-rhetorical parts of proofs and would have understood that her proof was not complete. As she had done on several previous occasions, Sofia suggested an idea that seemed to us not to be rationally connected to the problem at hand. We have come to call such actions 'unreflective guesses', and more colloquially, 'grasping at straws'. The

tutor then needed to take an action in response, as even doing nothing would likely have been interpreted by Sofia as an action or as having meaning. But to act usefully the tutor needed to understand why Sofia made her peculiar suggestion. We believe her suggestion was triggered by some kind of affect, but a kind not among the three major aspects previously discussed. On viewing the video, we found no evidence of an emotional response, and we have no reason to believe that Sofia had an underlying belief or attitude related to this or other similar incidents. We believe Sofia's suggestion was triggered by a feeling of confusion or by a feeling of not knowing what to do next.

The above example of Sofia suggests that it would be useful to expand one's view of affect somewhat, and that particular aspects of affect may be linked to specific parts of the proving process. DeBellis and Goldin [2–4] have added a forth main aspect to affect, namely, *values*. While this addition is a reasonable extension of affect, it also does not account for Sofia's behavior. DeBellis and Goldin further point out that 'affect is not auxiliary to cognition; it is centrally intertwined with it' [5]. They see 'affect as a highly structured system that encodes information, interacting fundamentally – and reciprocally – with cognition' [3, p. 2-249]. We now add a fifth aspect to affect, namely, *feelings*.

2.2 Feelings

DeBellis and Goldin, as well as Ortony, Clore, and Collins [8], see feelings as having an appraisal value that can be either positive or negative. In addition, Clore has considered 'feelings-as-information'. While DeBellis and Goldin [4] refer to 'emotional feelings,' Damasio [9] clearly distinguishes between emotions and feelings, with the former being public and the latter being private (p. 27). 'Emotions play out in the theatre of the body. Feelings play out in the theatre of the mind' (p. 28). Most of the emotions that Damasio considers, such as joy and sorrow, as well as some less intense emotions, are a complex collection of chemical and neural responses to a stimulus that may produce bodily changes, such as changes in one's heart rate, temperature, and so forth (p. 53).

Because this distinction between feelings and emotions may be somewhat counterintuitive, we paraphrase one of Damasio's salient examples. As doctors were placing tiny electrodes in the mesencephalon of the brain stem of a 65-year old woman suffering from Parkinson's disease, the patient abruptly stopped her ongoing conversation, began to look sad (evidence of emotion), and a *few seconds later* suddenly began to cry. She then said she had no energy to go on living (evidence of a feeling). The doctors quickly removed the offending electrode and the sobbing stopped as abruptly as it had begun, and the sadness vanished from the woman's face.

> The sequence of events in this patient reveals that the emotion sadness came first. The feeling of sadness followed … Once the stimulation ceased these manifestations waned and then vanished. The emotion disappeared and so did the feeling. … The importance of this rare neurological incident is apparent. … As thoughts normally causative of emotions appear in the mind, they cause emotions, which give rise to feelings, which conjure up other thoughts that … amplify the emotional state. … More emotion gives rise to more feeling, and the cycle continues until distraction or reason put an end to it. … By the time all these sets of phenomena are in full swing … it is difficult to tell by

introspection which came first. This woman's case helps us see through the conflation [between emotions and feelings]' [8, p. 57].

Thus, emotions and feelings are distinct, with the former being physical and the latter being mental. Each can sometimes elicit the other, and a particular feeling may, or may not, become lastingly associated with a particular emotion.

In discussing feelings, Clore [10] further distinguishes emotional feelings from nonemotional cognitive feelings, such as a feeling of knowing. For example, one might experience a feeling of knowing that one has seen a theorem useful in a current problem, but not be able to bring it to mind immediately. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it [10, p. 151].

Some nonemotional cognitive feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan [11] distinguishes the two. Of the former, he says that the 'intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before', and this feeling is different from a feeling of rightness [11, section 1, paragraph 3]. It is rightness, not familiarity that is 'the feeling-of-knowing in implicit cognition' [11, abstract]. Rightness is 'the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge' [11, section 1, paragraph 11].

Some feelings are not only nonemotional, but they also have no sensory component. They are non-sensory experiences – they are, for example, not red or hot. Such feelings do not have a verbal component, but are sometimes described in words. 'The feeling of familiarity is not a color, not an aroma, not a taste, not a sound. It is possible for the feeling of familiarity to merge with, or be absent from, virtually any sensory content found on any sensory dimension' [11, section 1, paragraph 7]. From the point of view of problem solving or proving, an important non-sensory experience is a feeling of rightness [11,12].

In regard to a feeling of rightness, Mangan [11, section 6, paragraph 7] says 'people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about 'gut feelings', 'just knowing', hunches, intuitions'.

We will focus on feelings that are often not intense, such as feelings of knowing, of caution, of familiarity, of confusion, of not knowing what to do next, of rightness/appropriateness, of rightness/direction, or of rightness/summation. Such feelings are non-sensory experiences that, at any given moment, can pervade one's whole conscious field. However, they may be rather 'vague' and not easily noticed or focused upon, but can influence one's actions [11, section 1, paragraph 4], including actions that are part of the proving process.

Feelings of rightness can give direction, be summative, or suggest appropriateness. Of a feeling of rightness/direction, Mangan says, 'In trying to solve, say, a demanding math problem, [a feeling of] rightness/wrongness gives us a sense of more or less promising directions long before we have the actual solution in hand' [11, section 6, paragraph 3].

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A feeling of rightness/summation can integrate and evaluate 'large sets of information necessary for the problem-solving [or theorem proving] processes' [9, p. 177]. For example, often at the end of reading or writing a proof, short-term memory is inadequate to hold sufficient detailed information to allow a conscious rational judgment of whether the proof is correct. However, something must cause an individual to decide his or her own, or someone else's, proof is correct. We see a (summative) feeling of rightness as playing a major role in such decisions.

2.3 Our perspective on affect

We see affect as having five main aspects: beliefs, values, attitudes, feelings, and emotions. Because a person *has,* or *experiences,* these aspects of affect, we regard them as part of what one might call the passive mind. In contrast, cognition, and in particular the proving process, is part of what one might call the active mind, referring to *mental acts*, for example, producing inner speech or vision, or bringing something to mind. Beliefs, values, and attitudes, like knowledge, are lasting in varying degrees, and can be 'activated', or brought to mind, but they are not themselves continually experienced, that is, they are not usually part of one's consciousness. In contrast, feelings and emotions are experiences, that is, they are part of one's consciousness. We follow Damasio [9] in taking emotions to be physically embodied and feelings to be mental states. Within feelings, some are associated with the senses, for example, a feeling of pain, and some are non-sensory, for example, the feeling that a certain argument is correct. Also, some feelings are associated with emotions, for example, a feeling of joy or sorrow; and some are not, for example, a feeling that an argument is correct.

Finally, although all of affect can provide information for cognition, certain non-sensory feelings are cognitive in that they are about, or relate to, parts of the cognitive process. For example, the feeling in constructing a proof that one is 'on the right track' is a cognitive feeling. In this paper, we focus on nonemotional cognitive feelings and how they might interact with the proving process.

3. The difficulty of obtaining information about feelings

Although a person's feelings are not directly observable, they are conscious and potentially reportable. However, not everything experienced is likely to be reported, for example, in 'think aloud' episodes. This is because consciousness has different aspects [11,12], some of which are not easily focused upon in order to formulate a report.

In discussing consciousness, we are referring to phenomenological consciousness, that is, to the subjective experiences that everyone is aware of. For example, this includes experiences arising from the senses (vision, hearing, etc.), one's own speech, and other physical actions, as well as the corresponding inner versions (inner vision, inner speech, etc.). In addition, consciousness includes more subtle experiences, often without a sensory component, such as the feeling that a proof is correct or that one is 'on the right track'.

At any given moment, a person's conscious field can accommodate a number of experiences, but this capacity is quite limited. Perhaps because of this limited capacity, some sensory experiences are focused upon and are of high resolution, and many other sensory

experiences are more peripheral, vague, and of lower resolution. If new sensory experiences are focused upon, then the earlier focal sensory experiences become more peripheral and vague. In contrast, non-sensory experiences pervade the entire conscious field, but are also often vague and of low resolution [11,12].

In think-aloud problem solving, the reporting process has a more limited capacity than consciousness, and there is no time for reflection, so reports are likely to be restricted mainly to what is being focused upon – probably the experiences of the problem solver's own immediate actions in the problem-solving process. Except in rare cases of very intense feelings, we expect feelings would only be mentioned in response to direct questions. Hurlburt, in studying consciousness, has developed a 'beeper' technique for obtaining 'snapshot' reports of the contents of a person's consciousness [13], but here again we suspect direct questions would be required to get detailed reports on feelings. Further, 'beeping' a student in the middle of proving a theorem would probably cause the student to lose his or her train of thought. Thus, one often finds oneself in the position of having to infer feelings.

4. Consciousness, the proving process and behavioral schemas

The psychological process of constructing a proof of a theorem is much more complex than the resulting proof, and no doubt partly occurs outside of consciousness. Here we suggest a theoretical perspective for a part of the proving process.

We see (at least the conscious part of) cognition in general, and the proving process in particular, as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions, or trying to remember an example. Many such actions appear to be guided by small 'habits of mind' that often link a particular recognized situation to a particular action. Such <situation, action> pairs, or habits of mind, can reduce the burden on working memory.

As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant become automatically linked to triggering situations. From a third-person, or outside, perspective these regularly linked <situation, action> pairs might be regarded as small 'habits of mind' [14]. On the other hand, taking a first-person, inside, or psychological perspective, they are lasting mental structures that we have called *behavioral schemas*. Enacting them depends on conscious awareness of a situation, but does not require conscious processing before the action. Such schemas are immediately available, that is, do not have to be recalled before use [15].

We suggest that behavioral schemas in general, and in particular those involved in proof construction, are a form of procedural knowledge. They can be acquired or learned through repeated similar proof constructing experiences, and their enactment can reduce the burden on a prover's working memory. Some schemas are beneficial and contribute to constructing correct proofs – they might be regarded as abilities.

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4.1 An illustration of a behavioral schema

Mary, an advanced graduate student in mathematics described to us a <situation, action> pair in proving that a compact subset A of R^n is bounded. Mary and two fellow graduate student assumed A to be unbounded and were able to construct an open cover of A that had no finite subcover. They immediately observed, without further reflection, that this contradicted the compactness of A and that this proved the result. Mary, who had never studied formal logic, reported that, upon finding the cover had no finite subcover, she immediately knew the result was proved. She did not reflect on the logical structure of what had transpired in an effort to explicitly warrant that the proof was complete. The other two students also did not appear to require reflection or an explicit warrant.

We infer that each student had recognized the situation as similar to one that they had experienced many times previously involving a hypothesis, a conclusion assumed false, and a resulting contradiction. The mental action was simply deciding the result had been proved. The link between the situation and the mental action appeared to be automatic and not to require reflection or a warrant. We see this as due to the students' extensive proof constructing experiences. However, many less experienced students require considerable reflection and wonder needlessly what should be contradicted. We see such <situation, action> pairs as common and as playing a significant role in the proving process.

4.2 The genesis and enactment of behavioral schemas

We regard <situation, action> pairs that occur regularly as the enactment of enduring mental structures that we call *behavioral schemas*. These are partly procedural knowledge, that is, knowing *how* to act, and are similar to what Mason and Spence [16] have called 'knowing *to* act in the moment'. We view behavioral schemas as belonging to a person's knowledge base. The action produced by the enactment of a behavioral schema might be simple, as in the above example, or complex, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema. When viewed in a large grain-size, behavioral schemas might also be regarded as habits of mind [14]. Habits of mind are similar to physical habits, but people are even less likely to be aware of them.

It appears that consciousness plays an essential role in understanding the enactment of behavioral schemas. This is reminiscent of the role consciousness plays in reflection. It is hard to see how reflection, treated as re-presenting experiences [17], could be possible without first having had the experiences, that is, without first being conscious of them. We offer the following six-point theoretical sketch of the genesis and enactment of behavioral schemas.

(1) Within very broad contextual considerations, behavioral schemas are immediately available. They do not normally have to be remembered, that is, searched for and brought to mind.

(2) Behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action.

(3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs.

(4) A behavioral schema that would produce a particular action cannot pass that information, outside of consciousness, to be acted on by another behavioral schema. The first action must actually take place and become conscious in order to become information acted on by the second behavioral schema. That is, one cannot 'chain together' behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action. For example, if the solution to a linear equation would normally require several steps, one cannot give the final answer without being conscious of some of the intermediate steps.

(5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person's actions, but such influence is normally not precise enough for doing mathematics.

(6) Behavioral schemas are acquired (learned) through habituation. That is, to acquire such a schema a person should carry out the appropriate action correctly a number of times. Changing an detrimental behavioral schema requires similar, perhaps longer, practice.

Elaborations of, and justifications for, this theoretical sketch can be found in [15].

5. The course

The setting from which our data are taken is a design experiment, consisting of a Modified Moore Method course [18,19], whose sole purpose is to improve the proving skills of beginning graduate and advanced undergraduate mathematics students. The course is consistent with a constructivist point of view, in that we attempt to maximize students' proof writing experiences. It is also somewhat Vygotskian in that we represent to the students how the mathematics community writes proofs.

The students are given self-contained notes consisting of statements of theorems, definitions, and requests for examples, but no proofs. The students present their proofs in class, and the proofs are critiqued. Suggestions for improvements in their notation and style of writing are also given. There are no formal lectures, and all comments and conversations are based on students' work. The course carries three credits and lasts one semester. It meets for one hour and fifteen minutes twice a week, making 30 class meetings per semester. We have now taught three, of a projected eight iterations of the course. There are two versions of the course, and either or both can be taken for credit. One version covers some basic ideas about sets, functions, real analysis, and semigroups. The other version covers sets, functions, some real analysis, and topology. The specific topics covered are of less importance than giving students opportunities to experience as many different types of proofs as possible.

6. Feelings and the proving process

6.1 Nonemotional cognitive feelings and the enactment of behavioral schemas

In constructing a proof, when a person recognizes a situation and enacts a behavioral schema that yields a mental action, a feeling of rightness or appropriateness may soon be experienced. For example, such a feeling can be generated internally by finding a warrant for the action while reflecting on the proof in an effort to establish its correctness. Such a feeling can also be generated by an external authority, such as a teacher certifying that the proof is correct. If the feeling occurs several times when the same behavioral schema is enacted, then the appropriateness of enacting the behavioral schema may be enduringly associated with the situation. We suggest that such an association of a positive feeling with a situation will increase the probability of enacting the schema in future encounters with the situation.

6.2 Edward's and Mary's reactions to considering fixed, but arbitrary elements

6.2.1 Edward

Students are often reluctant to consider an arbitrary, but fixed element in their proofs. We conjecture this is because they do not feel it right or appropriate to do so. For example, on the sixteenth day of the course, just over halfway through the Fall 2007 semester, Edward was proving that the identity function, *f*, on the set of real numbers is continuous. We were using the following definition: A function *f* is *continuous* at *a* means for all ε>0, there is a δ>0 so that for all *x*, if |*x*-*a*|<δ then |*f*(*x*)-*f*(*a*)|<ε.

In order to write a proof, Edward needed to link 'For all ε>0' in the definition of continuity (the situation) to writing in the proof something like 'Let ε be a number greater than 0', meaning ε is arbitrary, but fixed (the action). Edward was aware of the warrant for this action, namely, that because ε was arbitrary, anything proved about it must be true for every ε. At least eleven times previously in class, Edward had seen this kind of <situation, action> pair, including its warrant. For example, on the seventh day of the course, Edward had proved $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ essentially correctly, by choosing an element *x* in one side and proving *x* is an element in the other side, and vice-versa.

Here is Edward's proof that the identity function, *f*, is continuous on the set of real numbers. *Proof:* Let $a \in R$ and $\varepsilon > 0$ and $\delta > 0$ where $\delta = \varepsilon / 2$. So $\delta < \varepsilon$. For any $\varepsilon > 0$ and any $x \in R$ such that $|x - a| < \delta$. We have that $|f(x) - f(a)| < \delta < \varepsilon$ therefore *f* is continuous.

In the critique that followed, the teacher said it was inappropriate to write 'For any $\varepsilon > 0$' after writing 'Let $a \in R$ and $\varepsilon > 0$ …' because that means ε represents a particular number. Edward responded, 'It doesn't really matter [meaning the 'for any'] because I had $\varepsilon > 0$ there [meaning at the beginning of his proof].' Our interpretation is that Edward didn't really have a feeling of rightness/appropriateness for this <situation, action> pair, even after considerable experience with it. Furthermore, Edward made similar comments several times thereafter.

6.2.2 Mary

In a similar situation, Mary, the above mentioned advanced mathematics graduate student (section 4.1 above) only slowly developed a feeling of rightness/appropriateness regarding fixed, but arbitrary elements. We conducted interviews with Mary, and her real analysis teacher, Dr. K, about

events that had taken place two years earlier, when Mary was taking both a pilot version of our proofs course (section 5 above) and Dr. K's real analysis course. In the homework for Dr. K's course, Mary needed to prove many statements that included phrases like 'For all real numbers x,' where x represented a variable (the situation). In her proofs, Mary needed to write something like 'Let x be a number,' where x represented an arbitrary, but fixed number (the action). Dr. K often discussed Mary's proofs with her, and in particular, thought she carried out this action based on his authority.

When Mary was interviewed about this <situation, action> pair she said the following:

Mary: At that point [early in Dr. K's real analysis course] my biggest idea was, well he said to 'do it', so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.

Int: Did you have any feeling … if it was positive or negative, or extra …

Mary: Well, I guess I had a feeling of discomfort …

Int: Did this particular feature [having to fix x] keep coming up in proofs?

Mary: … it comes up a lot and what happened, and I don't remember [exactly] when, is

that instead of being rote and kind of uncomfortable, it started to just make sense

… By the end of the semester this was very comfortable for me.

We infer Mary developed both the behavioral schema and the associated feeling of appropriateness only after executing the <situation, action> pair numerous times. In early executions of this <situation, action> pair, Mary carried it out mainly based on Dr. K's authority. In addition, after completing each such proof, Mary attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof. Only after repeatedly executing this <situation, action> pair, and convincing herself that the individual proofs were correct, did she develop a feeling of appropriateness.

6.3 Students who focus too soon on the hypotheses

In a paper reporting on a mid-level undergraduate transition-to-proof course, Moore [20] described students' attempts to prove: *Let f and g be functions on A. If $f \circ g$ is one-to-one, then g is one-to-one.* This was a final examination question and 'all but one student [of 16] incorrectly attempted to begin [the problem-centered part of] the proof with the hypothesis – $f \circ g$ is one-to-one – rather than [starting to prove the conclusion] by assuming $g(x) = g(y)$ for a fixed but arbitrary x and y in A.'

6.3.1 Willy

We have noted a similar, persistent difficulty when students who were unable to construct a complete proof are asked to provide as much of one as they could. After writing little more than the hypotheses, some students turned immediately to focusing on those hypotheses, after which they could not complete the proof.

For example, on the twenty-sixth day of the Spring 2008 course, Willy was asked to prove

Theorem 29: *Let X and Y be topological spaces and $f : X \rightarrow Y$ be a homeomorphism of X onto Y. If X is a Hausdorff space, then so is Y.* Because only ten minutes of class time remained, we asked Willy to 'do the set-up', that is, present the formal-rhetorical part of the proof. Willy had indicated that he had not yet proved the theorem. However, early on in the course, he had developed some ability to write the formal-rhetorical parts of simple proofs.

On the left side of the board, Willy wrote:

Proof.   Let X and Y be topological spaces.

Let $f : X \rightarrow Y$ be a homeomorphism of X onto Y.

Suppose X is a Hausdorff space.

$\bullet \bullet \bullet$

Then Y is a Hausdorff space.

Then, on the right side of the board, he listed:

homeomorphism

one-to-one

onto

continuous (f is open mapping)

and then looked perplexedly back at the left side of the board. Even after two hints to look at the final line of his proof, Willy said, 'And, I was just trying to just think, homeomorphism means one-to-one, onto, …' After some discussion about the meaning of homeomorphism, the first author said, 'There is no harm in analysing what stuff you might want to use, but there is more to do before you can use any of that stuff', meaning that the conclusion should be examined and unpacked first. Willy did not make further progress that day.

We inferred that Willy was enacting a behavioral schema in which the situation was having written little more than the hypotheses, and the action was focusing on the meaning and potential uses of those hypotheses before examining the conclusion. This schema often leads to difficulties, as it did for Willy. We conjectured that Willy and other students, who were reluctant to look at, and unpack, the conclusion felt uncomfortable about this, or perhaps felt it more appropriate to begin with the hypotheses and work forward. We wonder whether this might be an expectation built up from proving secondary school geometry theorems, where it often seems that one can prove theorems from the 'top down'.

We also conjectured that, had Willy not been distracted by focusing on the meaning of homeomorphism, he might have written more of the formal-rhetorical part of the proof. That is, he might have filled in the blank space of his proof with something like:

Let y_1 and y_2 be two elements of Y.

$\bullet \bullet \bullet$

Thus there are disjoint open sets U and V so that $y_1 \in U$ and $y_2 \in V$.

Writing the formal-rhetorical part of a proof exposes the 'real problem' in this theorem, something Willy might have found tractable. Indeed, by the next class meeting, he had constructed a proof in the way we had expected.

6.4 Sofia's reaction to not having an idea

In section 6.2, we reported an incident of access to a rare first-person account that illustrated Mary's joint development of a feeling of appropriateness and a behavioral schema. In contrast, in this section, we describe how *we* used our theoretical perspective about feelings and behavioral schemas *to infer* that Sophia's progress was blocked by multiple enactments of an 'unreflective guess' schema, and how our intervention appears to have weakened that schema and improved her prospects for later success in graduate school.

6.4.1 Sofia's 'unreflective guess' schema

Sofia, the above mentioned first-year graduate student (section 2.1 above), was taking the second iteration of our proofs course (section 5 above). Our analysis is based on third-person information, that is, observations and video recordings of Sofia's class participation and of her proof constructions during seven individual tutoring sessions with the third author.

Sofia was a diligent student; however as the course progressed, an unfortunate pattern in her proving attempts emerged. When she did not have an idea for how to proceed, she often produced what one might call an 'unreflective guess' only loosely related to the context at hand, after which she could not make further progress. We do not see the common practice of guessing a direction and then adjusting according to its fruitfulness as inappropriate. However, although we could sometimes speculate on the origins of Sofia's guesses, we could not see how they could reasonably have been helpful in making a proof, nor did she seem to reflect on, or evaluate, them herself.

We inferred that, in such unreflective guessing, Sofia was enacting a behavioral schema that depended on a feeling. She was recognizing a situation, that is, that she had written as much of a proof as she could, and had a feeling of not knowing what to do next. This situation was linked in an automated way to the action of just guessing *any* approach that usually was only loosely related to the problem at hand without much reflection on its likely usefulness. We also felt that the enactment of an 'unreflective guess' schema was very likely to increase Sofia's feeling of not knowing what to do next, and thus, lead to another unreflective guess, or to abandoning her proof attempt. We judged that without our addressing this unfortunate behavioral schema, Sofia would not make progress on the problem-centered parts of proofs.

We began to suspect Sofia might have a persistant difficulty when she volunteered to present the following 'proof' on the fifth day of class. Only her first and last lines could reasonably be part of a proof.

Thm 4: For any sets A, B and C, if $A \subseteq B$ then $(A \cap C) \subseteq (B \cap C)$.

Pf: Let A, B and C be sets
Suppose $x \notin A$, $x \in B$ and $x \in C$
Then $x \notin (A \cap C)$, but $x \in (B \cap C)$
Therefore $(A \cap C) \subseteq (B \cap C)$ [Sofia did not end her sentences with periods.]

This behavior persisted for some time. Our intervention consisted of trying, during tutoring sessions, to prevent Sofia from enacting the 'unreflective guess' schema by suggesting substitute actions. These included: draw a figure, look for inferences from the hypotheses, reflect on everything done so far, or do something else for a while.

6.4.2 Helping Sofia to replace her 'unreflective guess' schema

Here is an example of how the third author led Sofia, towards constructing a beneficial behavioral schema by suggesting substitute actions. This hour and a half tutoring session occurred in the middle of the Spring 2008 course, and was devoted to helping Sofia prove Theorem 20: *Let* $(X, \mathcal{U})$ *be a topological space and* $Y \subseteq X$. *Then* $(Y, \{U \cap Y \mid U \in \mathcal{U}\})$ *is a topological space (called the relative topology on* Y *).*

When Sofia indicated she did not know how to prove the theorem, the tutor, in an attempt to deflect implementation of her 'unreflective guess' schema, suggested that she write the first and last lines of a proof and draw a sketch. With guidance, she then unpacked what was to be proved into four parts (the four defining properties of a topology), and she proved the first one, that is, $Y \in \{U \cap Y \mid U \in \mathcal{U}\}$. The tutor then tried to encourage her reflection on a key point, saying, 'Now, if I didn't understand that, and I asked you why Y is in there? The thing you said up above [$X \cap Y \in \{U \cap Y \mid U \in \mathcal{U}\}$]] doesn't have any Y in it. You would have to tell me that, well, Y is equal to $X \cap Y$, OK no problem.' Sofia said, 'um hum.'

Then Sofia wrote, '(2) Since [the empty set] $\varnothing \in \mathcal{U}$' and after 30 seconds said, 'I'm stuck.' It became clear that she did not know, in general, how to show an object is in a set when the defining variable in the set is compound (for example, $U \cap Y$). The tutor then said to forget the theorem for a moment and turn to the natural numbers. He wrote $\{2n \mid n \in N\}$ and asked 'Is 6 an element of that set?' Sofia answered yes, and the tutor asked why. Sofia and the tutor then agreed that the answer was yes because $6 = 2 \times 3$ and $3 \in N$. The tutor reiterated that 6 had to be represented in the form $2n$, where n has the appropriate property, that is, $n \in N$. Using this as a model, Sofia was then able to show $\varnothing \in \{U \cap Y \mid U \in \mathcal{U}\}$. With guidance, but less guidance than before, she also proved the third and fourth parts of the proof.

Here, the tutor's guidance and suggestions, not only helped Sofia prove the theorem, they also facilitated the construction of a beneficial behavioral schema in which the situation is needing to show an object is in a set (where the defining variable is compound), and the mental action is forming the intention to represent the object in the proper form and show the defining property of the set is satisfied. Sofia's previous work in the course had suggested she understood the language of sets and could draw appropriate Venn diagrams. However, for her that understanding was not equivalent to knowing how to use her knowledge in constructing a proof of Theorem 20.

As the course ended, our intervention of directing Sofia to do something else, whether it be draw a diagram or review her notes, was beginning to show promise. For example, on the in-class final examination Sofia proved that if f, g, and h are functions from a set to itself, f is one-to-one, and $f \circ g = f \circ h$, then $g = h$. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-oriented parts of at least a few proofs by the end of the course, and suggests her 'unreflective guess' behavioral schema was weakened.

7. Conclusion

In this paper we have described a design experiment to develop a course whose sole purpose is to improve advanced mathematics students' proving skills. The course appears to be useful because many advanced students have difficulty constructing proofs, and students' proofs are used as a major component in assessments of their understanding of content courses, such as abstract algebra or real analysis.

We have discussed the nature of feelings, especially nonemotional cognitive feelings and treated them as a part of affect. Also, a theoretical framework for analysing student progress with proofs is beginning to emerge. For example, we have suggested it is useful to distinguish between the formal-rhetorical parts of proofs and the problem-centered parts.

We further introduced a theoretical perspective suggesting that much of the proving process depends on procedural knowledge in the form of small habits of mind or behavioral schemas, some of which are beneficial, while others tend to produce difficulties. The way feelings can both arise from, and contribute to, the enactment of behavioral schemas was pointed out. The example of Mary and Dr. K was then provided to show how a feeling of appropriateness can be associated with, and strengthen, a behavioral schema that is often enacted in a student's proving process.

Our observations of Willy's, and other students', reluctance to examine the conclusion, and their preference for immediately examining the hypotheses, led us to infer that they had a feeling of discomfort or inappropriateness regarding this action.

Also, our interpretation of Sofia's persistent proving difficulty, as that of an 'unreflective guess' schema, was based on our conjecture that she had a feeling of not knowing what to do next, so she did things or offered suggestions only vaguely related to the theorem at hand. This interpretation led to a beneficial intervention.

We suspect there is much more to be investigated about the way nonemotional cognitive feelings and behavioral schemas relate to each other and to the proving process, and how proving is learned and can be better taught. In addition, the extent of advanced students' proving difficulties, together with the extent of the use of student proofs for assessing their understanding could be further investigated.

Acknowledgements

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