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A PARTIAL CHARACTERIZATION OF THE COCIRCUITS OF A SPLITTING MATROID

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Abstract. This paper describes some of the cocircuits of a splitting matroid $M_{x,y}$ in terms of the cocircuits of the original matroid M.

1. Introduction

The matroid notation and terminology used here will follow Oxley [2]. In particular, the ground set and the collections of independent sets, bases, and circuits of a matroid M will be denoted by $E(M)$, $\mathcal{I}(M)$, $\mathcal{B}(M)$, and $\mathcal{C}(M)$, respectively. The fundamental circuit of an element e with respect to the basis B (see [2, p. 18]) will be denoted by $C(e, B)$.

Fleischner [1] introduced the idea of splitting a vertex of degree at least three in a connected graph and used the operation to characterize Eulerian graphs. For example, the graph $G_{x,y}$ in Figure 1 is obtained from G by splitting away the edges x and y from the vertex v. Raghunathan, Shikare, and Waphare [3] extended the splitting operation from graphs to binary matroids. One of their results [3, Theorem 2.2] can be used to define the splitting operation in a binary matroid in terms of circuits.

Definition 1.1. Let M be a binary matroid and suppose $x, y \in E(M)$. The splitting matroid $M_{x,y}$ is the matroid having collection of circuits $\mathcal{C}(M_{x,y}) = \mathcal{C}_0 \cup \mathcal{C}_1$ where
$\mathcal{C}_0 = \{C \in \mathcal{C}(M) \mid x, y \in C \text{ or } x, y \notin C\}$; and
$\mathcal{C}_1 = \{C_1 \cup C_2 \mid C_1, C_2 \in \mathcal{C}(M), C_1 \cap C_2 = \emptyset, x \in C_1, y \in C_2$; and there is no $C \in \mathcal{C}_0$ such that $C \subseteq C_1 \cup C_2\}$.

The next result, due to Shikare and Asadi [4], characterizes the bases of a splitting matroid $M_{x,y}$ in terms of the bases of the original matroid M.

Lemma 1.2. *Let M be a binary matroid and suppose $x, y \in E(M)$. Then $\mathcal{B}(M_{x,y}) = \{B \cup \{\alpha\} \mid B \in \mathcal{B}(M),\ \alpha \in E - B$ and the unique circuit contained in $B \cup \alpha$ contains either x or $y\}$.*

The results in the next section describe some of the cocircuits of $M_{x,y}$ in terms of the cocircuits of M. Recall that the cocircuits of a matroid M

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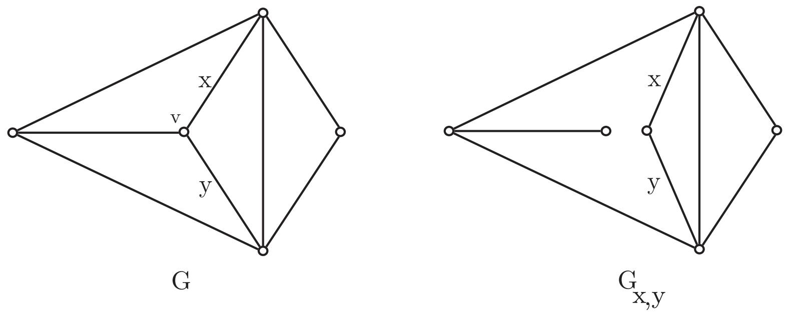

FIGURE 1. The graph $G_{x,y}$ is obtained by splitting vertex v of G.

are the minimal sets having non-empty intersection with every basis of M. In addition, the basic fact that if C is a circuit and C^* is a cocircuit of a matroid M, then $|C \cap C^*| \neq 1$ will be helpful in the proofs.

2. Cocircuits of a Splitting Matroid

It follows from Definition 1.1 that if every circuit of M contains both x and y, or neither, then $\mathcal{C}(M_{x,y}) = \mathcal{C}_0 = \mathcal{C}(M)$ and $M_{x,y} = M$. The fact that $M_{x,y} \neq M$ only if there is a circuit of M containing exactly one of x and y is the basis of the next two results.

Proposition 2.1. *If $\{x, y\}$ is a cocircuit of M or if $\{x\}$ and $\{y\}$ are cocircuits of M, then $M = M_{x,y}$.*

Proof. In both cases there is no circuit of M containing exactly one of x and y. Hence $M = M_{x,y}$. □

Proposition 2.2. *If exactly one of $\{x, y\}$ is a cocircuit of M, then x and y are cocircuits of $M_{x,y}$.*

Proof. Suppose x is a cocircuit of M and y is not. Then y is in a circuit of M that does not contain x. The circuits of $M_{x,y}$ either contain both x and y or contain neither x nor y. Since x is in no circuits of M, it follows from the definition of $\mathcal{C}(M_{x,y})$ that x is in no circuits of $M_{x,y}$. Thus y is in no circuits of $M_{x,y}$ and we conclude that y is a cocircuit of $M_{x,y}$. □

The previous two results concerned cases in which the set $\{x, y\}$ contained a cocircuit of M. The main result of this paper, Theorem 2.4, concerns the case in which $\{x, y\}$ is proper subset of a cocircuit of M. Before stating the main result, we first prove the following technical lemma.

Lemma 2.3. *Suppose C^* is a cocircuit of M and $\{x, y\} \subset C^*$. Then there exist bases B_1 and B_2 of M such that $B_1 \cap (C^* - \{x, y\}) = \emptyset$ and $B_2 \cap (C^* - \{x, y\}) = \emptyset$ where $\{x, y\} \cap B_1 = \{x\}$ and $\{x, y\} \cap B_2 = \{y\}$.*

Proof. Suppose C^* is a cocircuit of M and $\{x, y\} \subset C^*$. It follows from the minimality of C^* that there is a basis B of M so that $B \cap (C^* - \{x, y\}) = \emptyset$. Now suppose every basis of M having empty intersection with $C^* - \{x, y\}$ contains x. Then $C^* - y$ contains a cocircuit of M; a contradiction. Similarly, if each basis of M having empty intersection with $C^* - \{x, y\}$ contains y, then $C^* - x$ contains a cocircuit of M; a contradiction. We conclude that the lemma holds. □

Theorem 2.4. *Let $M_{x,y}$ be a splitting matroid obtained from M so that $M \neq M_{x,y}$. Suppose $\{x, y\}$ is a proper subset of a cocircuit C^* of M. Then $\{x, y\}$ and $C^* - \{x, y\}$ are cocircuits of $M_{x,y}$.*

Proof of Theorem 2.4. Suppose $\{x, y\}$ is a proper subset of a cocircuit C^* of M. We first show that $\{x, y\}$ is a cocircuit of $M_{x,y}$. Since $\mathcal{B}(M_{x,y}) = \{B \cup \alpha \mid B \in \mathcal{B}(M)$ and $C(\alpha, B)$ contains exactly one of x and $y\}$, it is clear that $\{x, y\}$ has non-empty intersection with each basis of $M_{x,y}$. Lemma 2.3 implies that there is a basis B of M so that $x \in B$, $y \notin B$ and $B \cap (C^* - \{x, y\}) = \emptyset$. Let $z \in C^* - \{x, y\}$. If $x \notin C(z, B)$, then $|C(z, B) \cap C^*| = 1$; a contradiction. Then $x \in C(z, B)$, and since $y \notin C(z, B)$, it follows that $B \cup z$ is a basis of $M_{x,y}$. Moreover, as $y \notin B \cup z$, the set $\{y\}$ is not a cocircuit of $M_{x,y}$. Similarly, $\{x\}$ is not a cocircuit of $M_{x,y}$. Since $\{x, y\}$ is a minimal set having non-empty intersection with each basis of $M_{x,y}$, the set $\{x, y\}$ is a cocircuit of $M_{x,y}$.

We now show that the set $C^* - \{x, y\}$ has non-empty intersection with each basis of $M_{x,y}$. Let $B \cup \alpha$ be an arbitrary basis of $M_{x,y}$. If $B \cap (C^* - \{x, y\}) \neq \emptyset$, then clearly $(B \cup \alpha) \cap (C^* - \{x, y\}) \neq \emptyset$. So we may assume B is a basis of M so that $B \cap (C^* - \{x, y\}) = \emptyset$. We complete this part of the proof by analyzing two cases. First, suppose $x, y \in B$. Now $B \in \mathcal{I}(M_{x,y})$ and $|B| < r(M_{x,y})r(M) + 1$. So B is a proper subset of a basis $B_1 \cup \alpha_1$ of $M_{x,y}$. Since $B_1 \cup \alpha_1 = B \cup \alpha$ for some α in $B_1 - B$, we may assume B is a proper subset of the basis $B \cup \alpha$ of $M_{x,y}$. Suppose $\alpha \in E(M) - (B \cup C^*)$. Then as $B \cup \alpha$ is a basis of $M_{x,y}$, the fundamental circuit $C(\alpha, B)$ in M must contain exactly one of x and y. This implies $|C(\alpha, B) \cap C^*| = 1$; a contradiction. We conclude that $\alpha \in C^* - \{x, y\}$. Hence $(B \cup \alpha) \cap (C^* - \{x, y\}) \neq \emptyset$

Now suppose $x \in B$ and $y \notin B$. Since $B \in \mathcal{I}(M_{x,y})$ and $|B| < r(M_{x,y}) = r(M) + 1$, there exists α in $E(M) - B$ so that $B \cup \alpha \in \mathcal{B}(M_{x,y})$. If $C(y, B)$ does not contain x, then $|C(y, B) \cap C^*| = 1$; a contradiction. Thus $C(y, B)$ contains both x and y. It follows that $B \cup y \notin \mathcal{B}(M_{x,y})$. Similarly, if $\alpha \in E(M) - (B \cup C^*)$, and $x \in C(\alpha, B)$, then $|C(\alpha, B) \cap C^*| = 1$; a contradiction.

So $C(\alpha, B)$ contains neither x nor y and it follows that $B \cup \alpha \notin \mathcal{B}(M_{x,y})$. We conclude that $\alpha \in C^* - \{x, y\}$. Hence $(B \cup \alpha) \cap (C^* - \{x, y\}) \neq \emptyset$. Therefore each basis of $M_{x,y}$ must have non-empty intersection with $C^* - \{x, y\}$.

We now show that $C^* - \{x, y\}$ is a minimal set having non-empty intersection with all bases of $M_{x,y}$. Let B be a basis of M so that $x \in B$, $y \notin B$, and $B \cap (C^* - \{x, y\}) = \emptyset$. Let $z \in C^* - \{x, y\}$. If $C(z, B)$ does not contain x, then $|C(z, B) \cap C^*| = 1$; a contradiction. Thus $x \in C(z, B)$. Moreover, $y \notin C(z, B)$ and it follows that $B \cup z \in \mathcal{B}(M_{x,y})$. Since for all $z \in C^* - \{x, y\}$, the set $B \cup z$ is a basis of $M_{x,y}$, the set $C^* - \{x, y\}$ is minimal having non-empty intersection with each basis of $M_{x,y}$. We conclude that $C^* - \{x, y\}$ is a cocircuit of $M_{x,y}$. □

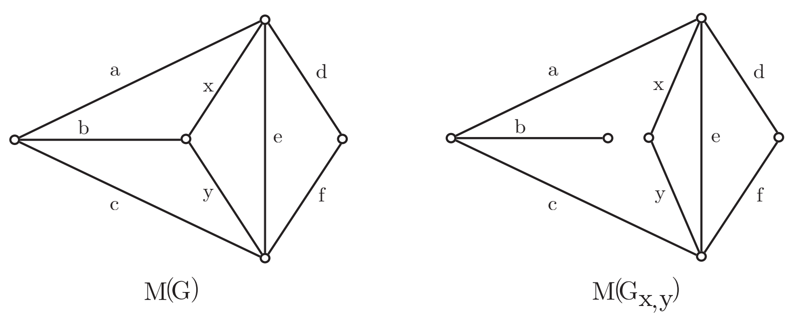

FIGURE 2. The matroids $M(G)$ and $M(G_{x,y})$.

Theorem 2.4 establishes that if C^* is a cocircuit of M containing $\{x, y\}$, then $C^* - \{x, y\}$ is a cocircuit of $M_{x,y}$. We define a Type I set of a matroid M to be a set $C^* - \{x, y\}$ where C^* is a cocircuit of M that properly contains $\{x, y\}$. The next table lists the collections of cocircuits of the matroids M and $M_{x,y}$ shown in Figure 2.

Cocircuits of M	Type I sets	Cocircuits of $M_{x,y}$
$\{d, f\}$	$\{b\}$	$\{d, f\}$
$\{a, b, c\}$	$\{a, c\}$	$\{b\}$
$\{b, x, y\}$		$\{a, c\}$
$\{a, x, y, c\}$		$\{x, y\}$
$\{c, y, e, f\}$		$\{a, y, e, d\}$
$\{c, y, e, d\}$		$\{a, x, e, d\}$
$\{a, x, e, d\}$		$\{a, y, e, f\}$
$\{a, x, e, f\}$		$\{a, x, e, f\}$
$\{b, x, e, d, c\}$		$\{c, x, e, d\}$
$\{b, x, e, f, c\}$		$\{c, y, e, d\}$
$\{a, b, y, e, f\}$		$\{c, y, e, f\}$
$\{a, b, y, e, d\}$		$\{c, x, e, f\}$

Notice that the cocircuits of $M_{x,y}$ are $\{x, y\}$, the Type I sets of M, the sets $D^* - X$ for each cocircuit D^* of M containing a Type I set X, and the cocircuits of M that do not contain a Type I set. The following conjecture proposes that this relationship holds in general.

Conjecture 2.5. *Suppose the splitting matroid $M_{x,y}$ is obtained from M and $\{x, y\}$ is a proper subset of a cocircuit of M. Then*

$$\mathcal{C}^*(M_{x,y}) = \begin{cases} \{x, y\} \\ C^* - \{x, y\} \text{ for each cocircuit } C^* \text{ of } M \text{ properly containing } \{x, y\} \\ D^* - X \text{ for each cocircuit } D^* \text{ of } M \text{ containing a Type I set } X \\ C^* \text{ of } M \text{ such that } C^* \text{ does not contain a Type I set} \end{cases}$$

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