DEPARTMENT OF MATHEMATICS

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CONSCIOUSNESS IN MATHEMATICAL PROBLEM SOLVING: THE FOCUS, THE FRINGE, AND NON-SENSORY PERCEPTION

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Consciousness in Mathematical Problem Solving: The Focus, the Fringe, and Non-Sensory Perception

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Introduction

In this paper, we will discuss the way various features of consciousness interact with each other and with cognition, specifically, the cognition of mathematical reasoning and problem solving. Thus we are interested in how consciousness and cognition "work," in a somewhat mechanistic way, rather than in larger philosophical questions about consciousness. Our goal is ultimately to answer questions like: Where does one's "next" idea come from? Answers to such smaller questions may eventually help in understanding the nature of consciousness itself.

We will discuss the relationship between consciousness and cognition in terms of two illustrations, and recall and extend some features of consciousness pointed out by Mangan (1993, 2001), following William James. The resulting framework will then be used do analyze three situations from mathematics: students (I) evaluating proofs, (II) writing equations, and (III) failing to use "adequate" knowledge to solve problems.

Some Features of Consciousness

Focus, Fringe, and Non-Sensory Perception

Consciousness has sometimes been thought of as partitioned into two parts -- the *focus* and the *fringe*, perhaps with a somewhat fuzzy boundary. Experiences in the focus have a higher resolution and are often more intense than those in the fringe. A fringe experience tends to be difficult to examine, perhaps because that very examination is likely to occasion a shift in focus to the experience, and hence, change its nature. The relationship between focus and fringe in consciousness is similar to the relationship between human central and peripheral vision, which has a clear physical basis. Similar relationships occur in other senses in some other animals.

Non-sensory experiences are often fringe experiences. They seem to develop across some time and are not the result of a change in one's current sensory input. For example, one might hear a musical passage a number of times without understanding or appreciating it. Then one might, rather suddenly, come to understand and appreciate the passage without its having changed. Furthermore, the *feeling of understanding and appreciation* will probably be remembered and re-experienced at appropriate times. Another example occurs in connection with carrying out a task. Some people experience a summative *feeling of rightness*, based on their having carried out the task properly, but without requiring any conscious examination of their actions. These ideas have been elucidated by Mangan (1993, 2001).

Extensions

1. We regard the focus and fringe structure (in both consciousness and vision) not simply as an "inexpensive" way to extend high resolution to a wide area, but more importantly, also as a way to guide one's future focus.

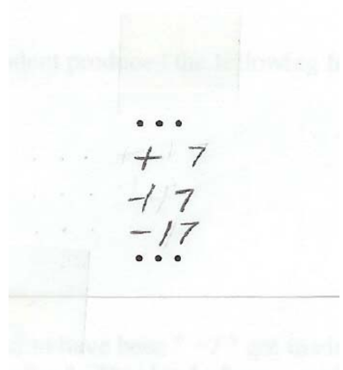
2. We see the development of non-sensory experiences as a form of learning and as possibly "linked" to other remembered experiences in a way that can also bring them into the fringe. Such links to remembered experiences provide objects for a possible future focus and this is a partial mechanism for "nonvoluntary" recall, i.e., coming to mind.

3. In addition to feelings of rightness (coherence, making sense, understanding), we are interested in *feelings of caution* and what we will call a *feeling of correctness* -- the feeling that one not only has made sense of, or understood, a mathematical proof, but also that the proof itself is logically correct. A feeling of correctness is more about the external world than about one's reaction to it.

Relationship Between Consciousness and Cognition

Illustration I

On a calculus test, a student produced the following fragment of a solution:



Here what was supposed to have been " +7 " got inadvertently converted into " -17 " through a poorly written " + ". This kind of error, which requires writing and reading, *could not* have been made outside of consciousness, or even in inner speech or vision. Thus, the inscription was not a simple record of more-or-less continuous mental work. The student appears to have "decided" how to write a line (probably outside of consciousness), written it, read it, and use that information (probably outside of consciousness) to "decide" what to write next. This supports that consciousness (of the reading), the act of writing, and even the inscription itself were integral parts of the student's cognition. We believe this is normal cognition which has only been made visible by the error. But was "deciding" what to write next, outside of consciousness? For this, consider the next illustration.

Illustration II

Suppose one is solving a simple, but moderately long, linear equations such as $3x+5=4x+2-7x$. This is usually done in several written steps. If, after writing a step and before writing the next, one's earlier work were suddenly and unexpectedly covered up, then one would very likely not be able to continue. Furthermore, between steps one is likely not to be conscious of anything happening. Clearly, however, something is happening (outside of consciousness) because the steps are not at all random.

We think that, for someone knowledgeable in algebra, the "decisions" needed to guide writing the various steps are made outside of consciousness and based on earlier steps that have become conscious. Furthermore, such "decisions" are ephemeral, and thus cannot be the basis for further "decisions," unless the earlier "decisions" are acted upon in a way that becomes conscious. Finally, the information needed to make such "decisions" seems to be very durable, always available, and need not become conscious to be usable. For example, that information might include: "It's OK to combine the $4x$ and the $-7x$ on the right."

Indeed, such information seems to be in some way "attached" to a recognition of the kind of problem at hand. This is very different from "normal" remembering which occasionally requires a search of one's knowledge base and seems to require conscious articulation before it can be used.

Summary

Cognition, looked at in a fine-grained way, often consists of brief periods of consciousness, alternating with ephemeral "decisions" made outside of consciousness and based on information from the previous conscious period plus durable immediately available information attached to one's view of the situation at hand. The "decisions" lead to actions (including mental actions) that become conscious and start the process anew. Since mathematical problem solving normally requires long chains of inferences, it depends on the above alternation of periods of conscious, nonconscious "decisions," and actions. Thus, it may not be quite appropriate to ask questions like: Is consciousness (treated as an independent phenomenon) necessary for cognition? At least in the situations we are examining, consciousness appears to be, not so much necessary for cognition, as an integral part of it.

We will now use the framework, or perspective, developed in the above discussion of the features of consciousness (focus, fringe, and non-sensory perception), and of the relationship between consciousness and cognition, to provide an at least plausible explanation for three puzzling situations from research in mathematics education.

Situation I: Reading Proofs

Deciding whether a mathematical proof is correct is a, usually private, complex mental process involving, for example, asking and answering questions, assenting to claims, and constructing subproofs. We call this process *validation*, although many mathematicians simply call it reading. Mathematicians can validate proofs remarkably reliably. However, this was far from the case for eight mid-level undergraduate mathematics and mathematics education majors we studied. Their judgments of four student-generated purported proofs of a single theorem (one right and three wrong) were only 46% correct, i.e., they might as well have flipped coins. However, in describing their previous experience in reading (correct textbook) proofs, the students sounded competent and emphasized reading for "understanding" (Selden & Selden, 2003).

When both mathematicians and students come to the end of a purported proof "something" tells them either the proof is correct or they should reexamine it for errors. For mathematicians, we suggest this "something" is either a *feeling of correctness* (which differs from a feeling of understanding, by including logical correctness of the proof) or a *feeling of caution*. In contrast, the students we studied appeared to be using their *feeling of understanding*, which served them poorly.

The validation of proofs is *not* often explicitly taught, but perhaps ought to be. We hope this analysis exposes an important pedagogical question: How does one teach a feeling?

Situation II: Writing Equations

In 1980 Rosnick and Clement introduced the Students-and-Professors Problem:

> *Write an equation using the variables S and P to represent the following statement: "There are six times as many students as there are professors at this university." Use S for the number of students and P for the number of professors.*

It turned out that many subjects (~40% of freshman engineering students), who "ought" to be able to correctly solve this problem easily, did not do so, incorrectly saying $6S=P$. Furthermore, since 1980, there have been at least 11 studies attempting to explain this anomaly. However, the resulting articles mainly document that no one really understands why so many people do not write this simple equation correctly. It has now been established that *none* of the explanations, such as the errors result from *syntactic translation*, are adequate (MacGregor & Stacey, 1993).

Using the framework developed here, we will suggest how a mathematician might solve the Students-and-Professors Problem correctly. The mathematician would be likely to recognize this problem as one of a familiar, but nameless, private class of algebra problems that are in danger of being set up incorrectly. This recognition, if it is conscious at all, would probably be in the fringe. The recognition would be attached to, or automatically generate, a *feeling of caution* that, in turn, would be attached to a checking method. (In this case, some small numbers might be substituted for the variables S and P.) He/She would then check his/her "first approximation" equation, and if the result were implausible, reverse it.

We suspect that many of the reported errors are due to people not having sufficient experience to have established a conception of algebra problems that are in danger of being set up incorrectly. Thus, they could not recognize the Students-and-Professors Problem as such a problem. They would have nothing to link to, or generate, a feeling of caution, and hence, would not experience it and have no reason to check their initial attempt at writing an equation.

Of course, this still does not answer the deeper psychological question of why there are algebra problems that are in danger of being set up incorrectly.

Situation III: Solving Problems

We have studied the ability of university calculus students to solve five *moderately nonroutine problems*, that is, problems moderately similar to, but not exactly like, problems they had been taught to solve (Selden, Selden, Hauk, & Mason, 2000; Selden, Selden, & Mason, 1994; Selden, Mason, & Selden, 1989). Such problems are important because there is no way to teach all, or even most, problems that can occur in the real world. Very few of even the most successful students could solve even one of the five problems, and taking additional calculus/differential equations classes helped only a little. Furthermore, often students who did not solve a problem could be seen, in a subsequent test, to have had *adequate knowledge* to solve it. For these students, the appropriate knowledge apparently was not lacking, but *did not come to mind*. Such students do not seem to think of various ways to begin a solution.

In contrast, calculus teachers do not have this difficulty. If they are asked to solve a problem, they soon have a method (that may, or may not, work), and if someone does not like that method, they will quickly find another. We suggest this is not just because teachers know more than students.

Typically, calculus teachers are asked to solve many unexpected problems. They can *recognize* many (unnamed) *kinds of problems*, e.g., derivative problems, problems with several equations, mainly algebra problems. We suggest these kinds of problems are mentally linked with various *tentative solution starts*. When a problem type is recognized (in the fringe), a process outside of consciousness brings one (or more) of the linked tentative solution starts to mind, at least in the fringe sense. The teacher can move his/her focus to this tentative solution start.

This process may not occur in students because although they, too, can recognized kinds of problems, these kinds of problems may not be mentally linked to tentative solution starts. *Why not?* Because students often solve problems for which the method is illustrated in textbook examples. Thus, they rarely must focus on how they will start a solution, and hence, rarely form the mental links between problem type and tentative solution starts.

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