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# A GENERALIZATION OF A GRAPH RESULT OF HALIN AND JUNG

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# A GENERALIZATION OF A GRAPH RESULT OF HALIN AND JUNG

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Abstract. This paper provides a partial generalization to matroid theory of the result of Halin and Jung that each simple graph with minimum vertex degree at least 4 has $K_5$ or the octahedron $K_{2,2,2}$ as a minor.

## 1. Introduction

The matroid notation and terminology used here will follow Oxley [4]. For a graph $G$, the associated simple graph will be denoted by $\widetilde{G}$. Similarly, the simple matroid associated with a matroid $M$ will be denoted by $\widetilde{M}$. We shall use $\delta(G)$ to denote the minimum vertex degree of a graph $G$. The purpose of this paper is to present an extension of the following result of Halin and Jung [1].

**Theorem 1.1.** *If $G$ is a simple graph such that $\delta(G) \geq 4$, then $G$ has a $K_5$- or $K_{2,2,2}$-minor.*

It is natural when attempting to extend a graph result concerning vertex degrees to matroid theory to allow cocircuit size to play the role of vertex degree in graph theory. We denote the minimum cocircuit size of a matroid $M$ by $g^*(M)$.

**Theorem 1.2.** *If $M$ is a 3-connected binary matroid such that $g^*(M) \geq 4$, then $M$ has a minor isomorphic to $M(K_{2,2,2})$, $M(K_5)$, $M^*(K_{3,3})$, or $F_7$.*

## 2. The Proof

The proof of Theorem 1.2 will use the following lemmas. The first is due to Hall [2].

**Lemma 2.1.** *If $G$ is a 3-connected graph, then $G$ has no $K_{3,3}$-minor if and only if either $G$ is planar or $\widetilde{G} \cong K_5$.*

The remaining three lemmas are results of Seymour [5] and they are restated as Proposition 11.2.3, Lemma 11.2.8, and Theorem 13.2.2 in [4].

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$$A_{10} = \left[\begin{array}{c|ccccc} & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 0 & 0 \\ I_5 & 0 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \\ & 1 & 0 & 0 & 1 & 1 \end{array}\right] \qquad A_{12} = \left[\begin{array}{c|cccccc} & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 1 & 0 \\ I_6 & 0 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

FIGURE 1. $GF(2)$ representations of $R_{10}$ and $R_{12}$.

**Lemma 2.2.** *If $M$ is a 3-connected binary matroid, then $M$ has no $F_7^*$-minor if and only if either $M$ is regular or $M \cong F_7$.*

The next lemmas involve the matroids $R_{10}$ and $R_{12}$. The matrices $A_{10}$ and $A_{12}$ shown in Figure 1 are $GF(2)$-representations of $R_{10}$ and $R_{12}$, respectively.

**Lemma 2.3.** *Let $e$ be an element of $R_{10}$. Then $R_{10}/e \cong M^*(K_{3,3})$.*

**Lemma 2.4.** *Let $M$ be a 3-connected regular matroid. Then either $M$ is graphic or cographic, or $M$ has a minor isomorphic to one of $R_{10}$ and $R_{12}$.*

Next we present the proof of Theorem 1.2.

*Proof.* Let $M$ be a 3-connected binary matroid such that $g^*(M) \geq 4$. Suppose $M = M^*(G)$ for some graph $G$ and has no minor isomorphic to $M^*(K_{3,3})$. Then $G$ has no minor isomorphic to $K_{3,3}$. It follows from Lemma 2.1 that either $G$ is planar or $G \cong K_5$. Thus $M$ is either graphic or $M \cong M^*(K_5)$. If $M$ is graphic then Theorem 1.1 implies that $M$ has an $M(K_5)$-minor or an $M(K_{2,2,2})$-minor. On the other hand, if $M \cong M^*(K_5)$, then $M$ has cocircuits of size 3; a contradiction. We conclude that the result holds if $M$ is cographic.

Now suppose $M$ is a 3-connected regular matroid and $g^*(M) \geq 4$. Then Lemma 2.4 implies that $M$ is either graphic or cographic, or has a minor isomorphic to $R_{10}$ or $R_{12}$. Since the result holds if $M$ is graphic or cographic, we may assume that $M$ has a minor isomorphic to $R_{10}$ or $R_{12}$. If $M$ has an $R_{10}$-minor then it follows from Lemma 2.3 that $M$ has an $M^*(K_{3,3})$-minor. We may now assume that $M$ has an $R_{12}$-minor. As the matroid $R_{12}$ is regular but not graphic, it follows that $R_{12}$ has a minor isomorphic to $M^*(K_5)$ or $M^*(K_{3,3})$. Since $R_{12}$ is self-dual, we conclude that it has an $M(K_5)$- or $M^*(K_{3,3})$-minor. Thus $M$ has such a minor.

Now suppose $M$ is a 3-connected non-regular binary matroid so that $g^*(M) \geq 4$. Then $M$ has an $F_7$- or $F_7^*$-minor. If $M$ has an $F_7$-minor then the result holds, so we may assume that $M$ has an $F_7^*$-minor. It follows from Lemma 2.2 that $M \cong F_7^*$. However $F_7^*$ has cocircuits of size 3; a contradiction. We conclude that the result holds for all 3-connected binary matroids. □

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## References

[1] Halin, R. and Jung, H. A., Über Minimalstrukturen von Graphen, insbesondere von $n$-fach zusammenhängenden Graphen, *Math. Ann.* **152** (1963), 75–94.
[2] Hall, D. W., A note on primitive skew curves, *Bull. Amer. Math. Soc.* **49** (1943), 935–937.
[3] Oxley, J. G., The binary matroids with no 4–wheel minor, *Trans. Amer. Math. Soc.* **301** (1987), 63–75.
[4] Oxley, J. G., *Matroid Theory,* Oxford University Press, New York, 1992.
[5] Seymour, P. D., Decomposition of regular matroids, *J. Combin. Theory. Ser. B* **28** (1980), 305–359.

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