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MATHEMATICS OF CLIFFORD -  
A MAPLE PACKAGE FOR CLIFFORD  
AND GRASSMANN ALGEBRAS

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# Mathematics of CLIFFORD - A Maple Package for Clifford and Graßmann Algebras

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## Abstract

CLIFFORD performs various computations in Graßmann and Clifford algebras. It can compute with quaternions, octonions, and matrices with entries in  $Cl(B)$  - the Clifford algebra of a vector space  $V$  endowed with an arbitrary bilinear form  $B$ . Two user-selectable algorithms for Clifford product are implemented: `cm1NUM` - based on Chevalley's recursive formula, and `cm1RS` - based on non-recursive Rota-Stein sausage. Graßmann and Clifford bases can be used. Properties of reversion in undotted and dotted wedge bases are discussed.

## 1. Introduction

As many programs, CLIFFORD emerged from a practical problem. Relatively complicated algebraic manipulations with octonions, which can be performed in `spin(7)`, started a thread which has now developed into a multi purpose algebra tool. It is the basic structure of a vector space  $V$  endowed with a quadratic form  $Q$  which is common to a vast host of mathematical, physical and engineering problems, and which, on the other hand, allows one to build naturally –i.e., in a categorial sense ‘for free’– an algebra structure, the Clifford algebra  $Cl(V, Q)$ . While in a conventional vector space framework one makes a good use of the vector space structure, it is tedious to compute with vectors since vector multiplication is lacking. Having established a Clifford algebra structure, it provides one with an entirely new formalism that now can be applied to solving new problems.

In this sense, CLIFFORD is a basic tool for all such investigations and applications which can be carried in finite dimensional vector spaces equipped with a

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quadratic form or, equivalently, with a symmetric bilinear form commonly called an *inner* or a *scalar product*. The intrinsic abilities of Maple even allow one to use CLIFFORD in projective and affine geometries to visualize complicated incidence relations that is so helpful, e.g., in image processing, visual perception, and robotics.

The authors of CLIFFORD have been interested in fundamental questions about  $q$ -deformed symmetries and quantum field theory. A reasonable number of new results has been derived by using systematically the ability to *compute* with a Computer Algebra System (CAS) at hand. Moreover, just asking questions as “what is the most general element fulfilling ...” has led to unexpected results and new insights. Testing of theorems, usually to check the consistency of the software, unfortunately has led from time to time to counter examples, that have made a rethinking and a reformulation necessary. However, the most striking ability of CLIFFORD is that it is unique in being able to handle Clifford algebras of an *arbitrary bilinear form* not restricted by symmetry and not directly related to a quadratic form. It is now well known that such structures are related to Hopf algebraic twists, and the later versions of CLIFFORD make a good use of a process called *Rota-Stein cliffordization* which turns out to be a Drinfeld twist of a Graßmann Hopf algebra (Brouder et al., 2002).

The present paper introduces the reader to the package. It is assumed that she is already familiar with Maple (Maple, 2003), a general purpose CAS; if not please consult, e.g., (Wright, 2002). Of course, such paper cannot be a *user guide* but may only be a demonstration of the usability and strength of the package. The interested reader is invited to download the package and to have a closer look at the online documentation in the Maple help browser or to download a .pdf file of approx. 500 help pages. Therein we provide also further mathematical background and references and a detailed description of every function. However, the present article provides also a quick –and dirty– introduction which should be sufficient for getting started.

A list of aims behind *experimental mathematics* which are the guiding beacons for this work is found in a companion paper (Abłamowicz and Fauser, 2003b) that describes a supplementary package BIGEBRA.

## 2. Notations and basic computations

CLIFFORD uses as default a standard Graßmann basis (Graßmann multivectors) in  $\bigwedge V$  where  $V = \text{span}\{\mathbf{e}_i \mid 1 \leq i \leq n\}$  for  $1 \leq n \leq 9$ . Then  $\bigwedge V = \text{span}\{\mathbf{1}, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \dots \wedge \mathbf{e}_k \mid 1 \leq i, j, k \leq n\}$ . In CLIFFORD these basis monomials are normally written as strings  $\{\text{Id}, e_1, \dots, e_9, e_1we_2, e_1we_3, \dots, e_1we_2we_3, \dots, \text{etc.}\}$  although they can be aliased to shorten input. Here  $e_1we_2$  is a string that denotes  $\mathbf{e}_1 \wedge \mathbf{e}_2$  and  $\text{Id}$  denotes the unity  $\mathbf{1}$  in  $\bigwedge V$ . For example, when  $n = 3$ , Graßmann basis monomials are:

```
> W=cbasis(3);
```

$$W = [\text{Id}, e_1, e_2, e_3, e_1we_2, e_1we_3, e_2we_3, e_1we_2we_3]$$

but aliases can also be used to shorten input/output:

```
> eval(makealiases(3));
```

$$I, e12, e21, e13, e31, e23, e32, e123, e132, e213, e231, e312, e321$$

In the above,  $e_{ijk} = e_i e_j e_k$  is the wedge product of three 1-vectors:  $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$ . Thus, the most general element of the Graßmann algebra  $\bigwedge V$  is a Graßmann polynomial which is just a linear combination of Graßmann basis monomials with real coefficients. Notice that symbolic indices are allowed:

```
> p1:=Id+4.5*ei-alpha*e1we2we3;
```

$$p1 := Id + 4.5 e_i - \alpha e_{123}$$

Reordering of Graßmann monomials can be explicitly accomplished by the procedure `reorder` while CLIFFORD procedures ordinarily return their results in the standard (reordered) basis.

```
> p2:=-e3we2we1-x0*Id+x12*e2we1+a*ejwei;reorder(p2);
```

$$p2 := -e_{321} - x_0 Id + x_{12} e_{21} + a e_{jwei} \\ e_{123} - x_0 Id - x_{12} e_{12} - a e_{iwej}$$

The wedge product  $\wedge$  is computed with the procedure `wedge` or its ampersand counterpart `&w`:

```
> wedge(e1,e2),e1 &w e2;wedge(ea,eb,ec),ea &w eb &w ec;p1 &w p2;
```

$$e_{12}, e_{12}$$

$$e_{awebwec}, e_{awebwec}$$

$$e_{123} - x_0 Id - 4.500000000 x_0 e_1 + \alpha x_0 e_{123} - x_{12} e_{12}$$

The Clifford product can be introduced in  $\bigwedge V$  by means of a left  $\lrcorner_B$  (or right  $\llcorner_B$ ) contraction dependent on an arbitrary bilinear form  $B : V \times V \rightarrow \mathbb{R}$  following Chevalley's recursive definition and it will be discussed in Sect. 3. This leads to elements of the Clifford algebra  $\mathcal{C}\ell(B)$  expanded into multivectors. Clifford multiplication is then implicitly dependent on  $B$ . It is given by the procedure `cmul` with the infix form `&c`.

```
> cmul(e1,e2),&c(e1,e2);cmul(ea,eb,ec);
```

$$e_{12} + B_{1,2} Id, e_{12} + B_{1,2} Id$$

$$e_{awebwec} + B_{b,c} e_a - B_{a,c} e_b + B_{a,b} e_c$$

Simultaneous computations in  $\mathcal{C}\ell(K)$  and  $\mathcal{C}\ell(B)$  can be performed since `cmul` accepts a name of a bilinear form as a parameter. For example,

```
> cmul[K](e1,e2),&c[K](e1,e2);cmul[K](ei,ej,ek);
```

$$e_{12} + K_{1,2} Id, e_{12} + K_{1,2} Id$$

$$e_{iwejwek} + K_{j,k} e_i - K_{i,k} e_j + K_{i,j} e_k$$

Of course, the form  $B$  or  $K$  can be numeric or symbolic. For example,

```
> B:=matrix(2,2,[1,a,a,1]);
```

$$B := \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$$

then the Graßmann basis for  $\mathcal{C}\ell(B)$  or  $\bigwedge V$  will be:

```
> cbas:=cbasis(2);
```

$$cbas := [Id, e_1, e_2, e_{12}]$$

while the Clifford multiplication table of the Graßmann basis monomials will be as follows:

```
> MultTable:=matrix(4,4,(i,j)->cmul(cbas[i],cbas[j]));
```

$$MultTable := \begin{bmatrix} Id & e1 & e2 & e12 \\ e1 & Id & e12 + a Id & e2 - a e1 \\ e2 & -e12 + a Id & Id & a e2 - e1 \\ e12 & a e1 - e2 & e1 - a e2 & (-1 + a^2) Id \end{bmatrix}$$

Of course, the Graßmann multiplication table will be:

```
> wedgetable:=matrix(4,4,(i,j)->wedge(cbas[i],cbas[j]));
```

$$wedgetable := \begin{bmatrix} Id & e1 & e2 & e12 \\ e1 & 0 & e12 & 0 \\ e2 & -e12 & 0 & 0 \\ e12 & 0 & 0 & 0 \end{bmatrix}$$

Let  $B = g + F$  where  $g(F)$  is the symmetric (antisymmetric) part of  $B$ :

```
> g,F:=matrix(2,2,[g11,g12,g12,g22]),matrix(2,2,[0,F12,-F12,0]);
B:=evalm(g+F);
```

$$g, F := \begin{bmatrix} g11 & g12 \\ g12 & g22 \end{bmatrix}, \begin{bmatrix} 0 & F12 \\ -F12 & 0 \end{bmatrix}$$

$$B := \begin{bmatrix} g11 & g12 + F12 \\ g12 - F12 & g22 \end{bmatrix}$$

Then, the multiplication table of the basis monomials in  $\mathcal{Cl}(B)$  will be:

```
> MultTable:=matrix(4,4,(i,j)->cmul(cbas[i],cbas[j]));
```

$$\begin{aligned} MultTable := & \\ & [Id, e1, e2, e12] \\ & [e1, g11 Id, e12 + (g12 + F12) Id, g11 e2 - (g12 + F12) e1] \\ & [e2, (g12 - F12) Id - e12, g22 Id, (g12 - F12) e2 - g22 e1] \\ & [e12, (g12 - F12) e1 - g11 e2, g22 e1 - (g12 + F12) e2, \\ & (g12^2 - F12^2 - g22 g11) Id - 2 e12 F12] \end{aligned}$$

Observe, that the “standard” relations  $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = (B_{i,j} + B_{j,i}) \mathbf{1} = 2g_{i,j} \mathbf{1}$  are satisfied by the generators  $\mathbf{e}_i, i = 1, 2, \dots, n$ , irrespective of the presence of the antisymmetric part  $F$  in  $B$ . For example,

```
> cmul[g](e1,e2)+cmul[g](e2,e1);
```

$$2 Id g12$$

```
> cmul[B](e1,e2)+cmul[B](e2,e1);
```

$$(g12 + F12) Id + (g12 - F12) Id$$

```
> clisort(simplify(%));
```

$$2 g12 Id$$

It is well known (Lounesto, 2001), (Lam, 1973) that real Clifford algebras  $\mathcal{Cl}(V, Q) = \mathcal{Cl}_{p,q}$  are classified in terms of the signature  $(p, q)$  of  $Q$  and the dimension  $\dim(V) = n = p + q$ . Information about all Clifford algebras  $\mathcal{Cl}_{p,q}, 1 \leq n \leq 9$  for any signature  $(p, q)$  has been pre-computed and stored in CLIFFORD. It can be retrieved with a procedure `clidata`.

```
> clidata([2,0]); #Clifford algebra of the Euclidean plane
```

$$[real, 2, simple, \frac{1}{2} Id + \frac{1}{2} e1, [Id, e2], [Id], [Id, e2]]$$

The above output indicates that the Clifford algebra  $Cl_2$  of the Euclidean plane  $\mathbb{R}^2$  is a simple algebra isomorphic to  $\text{Mat}(2, \mathbb{R})$  while the 4th entry in the data list is a primitive idempotent  $f$  that has been used to generate a minimal left spinor ideal  $S = Cl_2 f$  and, subsequently, the left spinor (lowest dimensional and faithful) representation of  $Cl_2$ . In general it is known that, depending on  $(p, q)$  and  $n = \dim(V)$ , the spinor ideal  $S = Cl_{p,q} f$  is a right  $K$ -module where  $K$  is either  $\mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$  for simple Clifford algebras when  $(p - q) \not\equiv 1 \pmod{4}$  or  $\mathbb{R} \oplus \mathbb{R}$  and  $\mathbb{H} \oplus \mathbb{H}$  for semisimple algebras when  $(p - q) \equiv 1 \pmod{4}$  (Lounesto, 1981), (Helmstetter, 1982). Elements in the third list  $[Id, e2]$  generate a real basis in  $S$  modulo  $f$ , that is,  $S = \langle Id \&c f, e2 \&c f \rangle = \langle f, e2 \&c f \rangle$ . Elements in the 5th list generate a subalgebra  $F$  of  $Cl(Q)$  that is isomorphic to  $K$ . In the case of  $Cl_2$  we have that  $F = \langle Id \rangle \cong \mathbb{R}$ . The last list gives  $2^k$  generators modulo  $f$  of  $S$  viewed as a right module over  $K$  where  $k = q - r_{q-p}$  and  $r$  is the Radon-Hurwitz number.<sup>†</sup> Number  $k$  gives a number of factors of type  $\frac{1}{2}\mathbf{1} + \frac{1}{2}T_i$ , where  $\{T_i\}, i = 1, \dots, k$ , are commuting Grassmann basis monomials that square to  $\mathbf{1}$  whose product gives a primitive idempotent  $f$  in  $Cl(Q)$ . For all Clifford algebras  $Cl(Q)$ ,  $1 \leq n = p + q \leq 9$ , and for any signature  $(p, q)$  of  $Q$ , spinor representations have been pre-computed (Abłamowicz, 1998) and can be retrieved with the procedure `matKrepr`. For example, 1-vectors  $e_1$  and  $e_2$  in  $Cl_2$  are given in the  $\langle f, e2 \&c f \rangle$  basis of  $S = Cl_2 f$  as:

```
> matKrepr([2,0]);
```

$$[e1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}]$$

In another example, Clifford algebra  $Cl_3$  of  $\mathbb{R}^3$  is isomorphic with  $\text{Mat}(2, \mathbb{C})$  :

```
> B:=linalg[diag](1,1,1):clidata([3,0]);
```

$$[complex, 2, simple, \frac{1}{2} Id + \frac{1}{2} e1, [Id, e2, e3, e23], [Id, e23], [Id, e2]]$$

and its spinor representation is given in terms of Pauli matrices:

```
> matKrepr([3,0]);
```

$$[e1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e3 = \begin{bmatrix} 0 & -e23 \\ e23 & 0 \end{bmatrix}]$$

Notice that  $F = \langle Id, e23 \rangle$  ( $e23 = e2we3$ ) is a subalgebra of  $Cl_3$  isomorphic to  $\mathbb{C}$ . Since Pauli matrices belong to  $\text{Mat}(2, F)$ , it is necessary for CLIFFORD to compute with Clifford matrices, that is, matrices of type `clmatrix` with entries in a Clifford algebra.

```
> M1,M2,M3:=rhs(%[1]),rhs(%[2]),rhs(%[3]);
```

$$M1, M2, M3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -e23 \\ e23 & 0 \end{bmatrix}.$$

<sup>†</sup>Type `?RHnumber` in a Maple session when CLIFFORD is installed for more help.

Of course Pauli matrices satisfy the same defining relations as the basis vectors  $e_1, e_2$ , and  $e_3$ :<sup>‡</sup> For example:

```
> 'M1 &cm M2 + M2 &cm M1' = evalm(M1 &cm M2 + M2 &cm M1);
'e1 &c e2 + e2 &c e1' = e1 &c e2 + e2 &c e1;
```

$$M1 \&cm M2 + M2 \&cm M1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e1 \&c e2 + e2 \&c e1 = 0$$

```
> 'M1 &cm M1' = evalm(M1 &cm M1), 'M2 &cm M2' = evalm(M2 &cm M2),
'M3 &cm M3' = evalm(M3 &cm M3);
'e1 &c e1' = e1 &c e1, 'e2 &c e2' = e2 &c e2, 'e3 &c e3' = e3 &c e3;
```

$$M1 \&cm M1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M2 \&cm M2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M3 \&cm M3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$e1 \&c e1 = Id, e2 \&c e2 = Id, e3 \&c e3 = Id$$

The procedure `matKrepr` gives the linear isomorphism between  $Cl(Q)$  and  $Mat(2, \mathbb{R})$ , and, in general, between  $Cl(Q)$  and  $Mat(2^k, K)$ ,  $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , for simple algebras and  $Cl(Q)$  and  $Mat(2^k, K) \oplus Mat(2^k, K)$ ,  $K = \mathbb{R}, \mathbb{H}$ , for semi-simple algebras. In this latter case, it is customary to represent an element in  $Cl(Q)$  in terms of a single matrix over a double field  $\mathbb{R} \oplus \mathbb{R}$  or  $\mathbb{H} \oplus \mathbb{H}$  rather than as pair of matrices.<sup>§</sup>

One can easily list signatures of the quadratic form  $Q$  for which  $Cl(Q)$  is simple or semisimple. For more information, type `?all_sigs`. For example,  $Cl_{1,3}$  has a spinor representation given in terms of 2 by 2 quaternionic matrices whose entries belong to a subalgebra  $F$  of  $Cl_{1,3}$  spanned by  $\langle Id, e2, e3, e2we3 \rangle$ :

```
> B:=linalg[diag](1,-1,-1,-1):clidata([1,3]);
```

$$[\textit{quaternionic}, 2, \textit{simple}, \frac{1}{2} Id + \frac{1}{2} e1we4, [Id, e1, e2, e3, e12, e13, e23, e123],$$

$$[Id, e2, e3, e23], [Id, e1]]$$

```
> matKrepr([1,3]); # quaternionic matrices
```

$$[e1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e2 = \begin{bmatrix} e2 & 0 \\ 0 & -e2 \end{bmatrix}, e3 = \begin{bmatrix} e3 & 0 \\ 0 & -e3 \end{bmatrix}, e4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}]$$

CLIFFORD has built-in several special-purpose procedures to deal with quaternions and octonions (type `?quaternion` and `?octonion` for help). In particular, following Lounesto et al. (1987), octonions are treated as para-vectors in  $Cl_{0,7}$  while their non-associative multiplication, defined via Fano triples, is related to the Fano projective plane  $\mathbb{F}_2$  (see `?omultable`, `?Fano_triples` for more information). Since the bilinear form  $B$  can be degenerate<sup>¶</sup>, one can use CLIFFORD to perform computations in  $Cl_{0,3,1}$ , the Clifford algebra of the quadratic form

<sup>‡</sup>Here `&cm` is a matrix product where Clifford multiplications is applied to the matrix entries. See `?&cm` for more information.

<sup>§</sup>Procedures `adfmatrix` and `mdfmatrix` add and multiply matrices of type `dfmatrix` over such double fields. For more information see `?matKrepr`.

<sup>¶</sup>When  $B \equiv 0$  then  $Cl(V, B) = \bigwedge V$  and computations in the Grassmann algebra  $\bigwedge V$  can then be done with CLIFFORD.

$Q(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2$  where  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4 \in \mathbb{R}^4$ . This algebra is used in robotics to represent rigid motions in  $\mathbb{R}^3$  and screw motions in terms of dual quaternions (Selig, 1996).

Thus, CLIFFORD is a repository of mathematical knowledge about Clifford algebras of a quadratic form in dimensions 1 through 9. Together with the supplementary package BIGEBRA, (Abłamowicz and Fauser, 2001) it can be extended to graded tensor products of Clifford algebras in higher dimensions. Package BIGEBRA (see ?Bigebra) is described in a companion paper (Abłamowicz and Fauser, 2003b). For more information about any CLIFFORD procedure, see its help page through the Maple browser. For a computation of spinor representations with CLIFFORD, see Abłamowicz (1998).

### 3. Clifford product

#### 3.1. Recursive procedure `cmulNUM`

Since the Clifford product provides the main functionality of the CLIFFORD package, the best available mathematics has been used in its code. The user normally does not use the internal functions `cmulRS` and `cmulNUM` but the wrapper function `&c[K](arg1, arg2, ...)` or, in long form, `cmul[K](arg1, arg2, ...)` to pass the name of a bilinear form  $K$ . The wrapper function can also act on any number of arguments of type `clipolynom` (since the Clifford product is associative this makes sense) and on a much wider class of types including Clifford matrices of type `climatrix`. It can also accept Clifford polynomials in other bases such as the Clifford basis  $\{\mathbf{1}, \mathbf{e}_i, \mathbf{e}_i \&\mathcal{C} \mathbf{e}_j, \&\mathcal{C}(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k), \dots\}$  where  $\&\mathcal{C}$  denotes the unevaluated Clifford product. Clifford basis differs from the Graßmann exterior basis when  $B$  is not a diagonal matrix.<sup>||</sup>

There are two internal Clifford multiplication procedures which are appropriate for different purposes: `cmulRS` and `cmulNUM`. While `cmulNUM` is fast on sparse numeric matrices and on numeric matrices in general for dimensions  $\dim(V) \geq 5$ , procedure `cmulRS` was designed for symbolic calculations. Since `cmulRS` computes reasonably well in the numeric sparse case up to  $\dim(V) = 5$ , it was chosen as the default product of the package. Both internal Clifford multiplication procedures take two Clifford monomials of type `clibasmon` as input together with a third argument of type `name`, `symbol`, `matrix` or `array` which represents the chosen bilinear form.

There is a facility to let the user select `cmulRS` or `cmulNUM` when knowledge of the bilinear form allows one to decide which procedure might yield better performance. The user can supply a new product function (not necessarily a Clifford product) acting on two basis monomials. The wrapper uses an appropriate function, `cmulRS`, `cmulNUM`, or the other, which can be selected via a special function `useproduct`.

<sup>||</sup>Procedures converting between Graßmann and Clifford bases are part of a supplementary package `Cl15plus` (Abłamowicz and Fauser, 2001) while Clifford polynomials expressed in the Clifford basis are of type `cliprod`. Type `?cliprod` for more information.

The evaluation of Clifford products in a Graßmann basis is quite involved and normally is done by a recursive process that involves Chevalley deformation. This approach has been employed in `cmulNUM`. However, Hopf algebraic methods can be applied also, and have been used in `cmulRS`. Unfortunately, during the recursive evaluation many unnecessary terms are calculated which later cancel out at the next recursive call. This feature, while being beneficial when the bilinear form is sparse numeric since it cuts out many branches of the recursion quite early, prevents fast evaluation in the symbolic case where, in general, all terms might be non-zero. From this observation, the two possibilities to evaluate the Clifford product have emerged.

We introduce the Chevalley deformation and the Clifford map to explain the algorithm used in `cmulNUM`. The Clifford map  $\gamma_{\mathbf{x}}$  is defined on  $u \in \bigwedge V$  as

- (i)  $\gamma_{\mathbf{x}}(u) = \text{LC}(\mathbf{x}, u, B) + \text{wedge}(\mathbf{x}, u) = \mathbf{x} \lrcorner_B u + \mathbf{x} \wedge u$
- (ii)  $\gamma_{\mathbf{x}}\gamma_{\mathbf{y}} = \gamma_{\mathbf{x} \wedge \mathbf{y}} + B(\mathbf{x}, \mathbf{y})\gamma_{\mathbf{1}}$
- (iii)  $\gamma_{a\mathbf{x}+b\mathbf{y}} = a\gamma_{\mathbf{x}} + b\gamma_{\mathbf{y}}$

where  $\mathbf{x}, \mathbf{y} \in V$  (Lounesto, 2001). One knows how to compute with the wedge  $\mathbf{x} \wedge u$  and the left contraction  $\text{LC}(\mathbf{x}, u, B) = \mathbf{x} \lrcorner_B u$  with respect to the bilinear form  $B$  (in `CLIFFORD` left contraction is given by the procedure `LC`). Following Chevalley, the left contraction has the following properties:

- (i)  $\mathbf{x} \lrcorner_B \mathbf{y} = B(\mathbf{x}, \mathbf{y})$
- (ii)  $\mathbf{x} \lrcorner_B (u \wedge v) = (\mathbf{x} \lrcorner_B u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner_B v)$
- (iii)  $(u \wedge v) \lrcorner_B w = u \lrcorner_B (v \lrcorner_B w)$

where  $\mathbf{x} \in V, u, v \in \bigwedge V$  and  $\hat{u}$  is Graßmann grade involution. Hence we can use the Clifford map  $\gamma_{\mathbf{x}}$  (Chevalley deformation of the Graßmann algebra) to define a Clifford product of a one-vector and a multivector. Analogous formulas can be given for a right Clifford map using the right contraction  $\llcorner_B$  implemented as the procedure `RC`.

The Clifford product `cmul` or its ampersand form `&c` can now be defined as follows: We have to split off recursively a single element from the first factor of the product of two Graßmann basis monomials and then use Chevalley's Clifford map. For example,

$$\begin{aligned} (\mathbf{e}_a \wedge \dots \wedge \mathbf{e}_b \wedge \mathbf{e}_c) \&c (\mathbf{e}_f \wedge \dots \wedge \mathbf{e}_g) = \\ (\mathbf{e}_a \wedge \dots \wedge \mathbf{e}_b) \&c (\mathbf{e}_c \lrcorner_B (\mathbf{e}_f \wedge \dots \wedge \mathbf{e}_g) + \mathbf{e}_c \wedge \mathbf{e}_f \wedge \dots \wedge \mathbf{e}_g) \\ - (\mathbf{e}_a \wedge \dots \wedge \mathbf{e}_b \llcorner_B \mathbf{e}_c) \&c (\mathbf{e}_f \wedge \dots \wedge \mathbf{e}_g). \end{aligned} \quad (1)$$

Specifically, for  $(\mathbf{e}_1 \wedge \mathbf{e}_2) \&c (\mathbf{e}_3 \wedge \mathbf{e}_4)$  we have

$$\begin{aligned} (\mathbf{e}_1 \wedge \mathbf{e}_2) \&c (\mathbf{e}_3 \wedge \mathbf{e}_4) &= (\mathbf{e}_1 \&c \mathbf{e}_2) \&c (\mathbf{e}_3 \wedge \mathbf{e}_4) - B(\mathbf{e}_1, \mathbf{e}_2)\mathbf{1} \&c (\mathbf{e}_3 \wedge \mathbf{e}_4) \\ &= \mathbf{e}_1 \&c (B(\mathbf{e}_2, \mathbf{e}_3)\mathbf{e}_4 - B(\mathbf{e}_2, \mathbf{e}_4)\mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4) \\ &\quad - B(\mathbf{e}_1, \mathbf{e}_2)\mathbf{1} \&c (\mathbf{e}_3 \wedge \mathbf{e}_4) \end{aligned}$$

and a second recursion of the process gives now

$$\begin{aligned}
&= B(\mathbf{e}_2, \mathbf{e}_3)B(\mathbf{e}_1, \mathbf{e}_4) - B(\mathbf{e}_2, \mathbf{e}_4)B(\mathbf{e}_1, \mathbf{e}_3) + B(\mathbf{e}_2, \mathbf{e}_3)(\mathbf{e}_1 \wedge \mathbf{e}_4) \\
&\quad - B(\mathbf{e}_2, \mathbf{e}_4)(\mathbf{e}_1 \wedge \mathbf{e}_3) + \mathbf{B}(\mathbf{e}_1, \mathbf{e}_2)(\mathbf{e}_3 \wedge \mathbf{e}_4) - B(\mathbf{e}_1, \mathbf{e}_3)(\mathbf{e}_2 \wedge \mathbf{e}_4) \\
&\quad + B(\mathbf{e}_1, \mathbf{e}_4)(\mathbf{e}_2 \wedge \mathbf{e}_3) + \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 - \mathbf{B}(\mathbf{e}_1, \mathbf{e}_2)(\mathbf{e}_3 \wedge \mathbf{e}_4)
\end{aligned}$$

with the bolded terms cancelling out. Note that the last term in the r.h.s. was superfluously generated in the first step of the recursion.

The Clifford product can be derived from the above recursion by linearity and associativity. The induction starts with a left factor of grade one or grade zero which is trivial, i.e.,  $\mathbf{1} \&c \mathbf{e}_a \wedge \dots \wedge \mathbf{e}_b = \mathbf{e}_a \wedge \dots \wedge \mathbf{e}_b$ . In the case when the left factor is of grade one, we use the Clifford product expressed by the Clifford map of Chevalley, i.e.,  $\mathbf{e}_a \&c \mathbf{e}_b \wedge \dots \wedge \mathbf{e}_c = \mathbf{e}_a \lrcorner_B \mathbf{e}_b \wedge \dots \wedge \mathbf{e}_c + \mathbf{e}_a \wedge \mathbf{e}_b \wedge \dots \wedge \mathbf{e}_c$ . We make a complete induction in the following way: If the left factor is of higher grade, say  $n$ , one application of the recursion yields Clifford products where the left factor side is of grade either  $n - 1$  or  $n - 2$ , hence the recursion stops after at most  $n - 1$  steps.

A disadvantage of the recursive approach is that additional terms are produced by shifting Graßmann wedge products into Clifford products to swap one factor to the right. These terms cancel out, but this process increases unnecessarily the computing time.

An advantage of the recursive approach occurs when the bilinear form  $B$  is numeric and sparse, that is, with many zeros. In this case after any recursion many terms drop out since Maple immediately simplifies such expressions and only a few remaining terms enter the next recursive step. If the dimension of  $V$  is large, i.e.,  $\dim(V) \geq 6$ , sparse matrices benefit drastically from this process over the Hopf algebraic approach of `cmulRS` which computes all terms without benefiting from the special sparse numeric form of the bilinear form.

One could think about shifting factors from right to left, however this works out in the same way. Moreover, if the grade of the left factor  $n$  is greater than the grade  $m$  of the right factor, then the recursion stops also (since the terms evaluate to zero) after at most  $n - 1$  steps, so no increase in performance can be gained this way. For example, the above computation in Maple will be performed as follows:

```
> cmul(e1we2, e3we4);
```

$$(B_{2,3} B_{1,4} - B_{2,4} B_{1,3}) Id + B_{2,3} e14 - B_{2,4} e13 - B_{1,3} e24 + B_{1,4} e23 + e1234$$

Notice also that `cmul` accepts an arbitrary bilinear form  $K$  as its argument:

```
> cmul[K](e1we2, e3we4);
```

$$(K_{2,3} K_{1,4} - K_{2,4} K_{1,3}) Id + K_{2,3} e14 - K_{2,4} e13 - K_{1,3} e24 + K_{1,4} e23 + e1234$$

and likewise its ampersand form\*\*

\*\*Procedures `cmulNUM` and `cmulRS` do not have their special ampersand forms. However, procedure `&c` uses internally one procedure or the other depending on the current value of an environmental variable `_default_Clifford_product`. Current values of these variables can be displayed by `CLIFFORD_ENV`.

> `&c[K](ei, ejwekwel);`

$$eiwejwekwel + K_{i,j} ekwel - K_{i,k} ejwel + K_{i,l} ejwek$$

where we have also shown the ability to use symbolic indices. For clarity and to show our approach we display the algorithm of `cmulNUM` in Appendix A.

### 3.2. Procedure `cmulRS` based on the Rota-Stein combinatorial process

The procedure `cmulRS` is computed using the non-recursive Rota-Stein cliffordization (Rota and Stein, 1994; Fauser, 2002; Ablamowicz and Fauser, 2003a,b) and the help pages of the `BIGEBRA` package for further literature. The cliffordization process is based on the Hopf algebra theory. The Clifford product is obtained from the Graßmann wedge product and its Graßmann co-product as follows (in tangle notation):

$$\text{\&c} := \text{tangle diagram} \tag{2}$$

where  $\wedge$  is the Graßmann exterior wedge product and  $\Delta_\wedge$  is the Graßmann exterior co-product, which is obtained from the wedge product by categorical duality, i.e., to every algebra over a linear space  $A$  having a product we find a co-algebra having a co-product over the same space by reversing all arrows in all axiomatic commutative diagrams. Note that the co-product splits each input ‘factor’  $x$  into a sum of tensor products of ordered pairs  $x_{(1)i}, x_{(2)i}$ . The main requirement is that every such pair multiplies back to the input  $x$  when the dual operation of multiplication is applied, i.e.,  $x_{(1)i} \wedge x_{(2)i} = x$  for each  $i$ th pair. The ‘cup’ like part of the tangle decorated with  $B^\wedge$  is the bilinear form  $B$  on the generating space  $V$  extended to the whole Graßmann algebra, i.e., the map  $B^\wedge : \bigwedge V \times \bigwedge V \rightarrow k$  with  $B : V \times V \rightarrow k$  as  $B(\mathbf{x}, \mathbf{y})$  on vectors  $\mathbf{x}$  and  $\mathbf{y}$ . Hence, `cmulRS` computes on Graßmann basis monomials  $x$  and  $y$  for the given  $B$ , later extended to polynomials by bilinearity, as follows:

$$\text{cmulRS}(x, y, B) = \sum_{i=1}^n \sum_{j=1}^m (\pm) x_{(1)i} \wedge y_{(2)j} B(x_{(2)i}, y_{(1)j}) \tag{3}$$

where  $n, m$  are dummies describing the cardinalities of the required splits and the sign is due to the parity of a permutation needed to arrange the factors.

The simplified algorithm of `cmulRS` is as follows:

---

```

1 > cmulRS(x,y,B) [x, y two Grassmann monomials, B - bilinear form]
2 > begin
3 >   lstx <- list of indices from x
4 >   lsty <- list of indices from y

```

```

5 > NX <- length of lstx
6 > NY <- length of lsty
7 > funx <- function maps integers 1..NX onto elements of lstx keeping their order
8 > funy <- function maps integers 1..NY onto elements of lsty keeping their order
9 > (this is to calculate with arbitrary indices and to compute necessary signs)
10 > psetx <- power set of 1..NX (actually a list in a certain order)
11 > (the i-th and (2^NX+1-i)-th element are disjoint adding up to the set 1..NX
12 > psety <- power set of 1..NY (actually a list in a certain order)
13 > (the i-th and (2^NY+1-i)-th element are disjoint adding up to the set 1..NY
14 > (for faster computation we sort this power sets by grade)
15 > (we compute the sign for any term in the power set)
16 > psetx <- sort psetx by grade
17 > psety <- sort psety by grade
18 > pSgnx <- sum_(i in psetx) (-1)^sum_(j in psetx[i]) (psetx[i][j]-j)
19 > pSgny <- sum_(i in psety) (-1)^sum_(j in psety[i]) (psety[i][j]-j)
20 > (we need a subroutine for cup tangle computing the bilinear form cup(x,y,B))
21 >   begin
22 >     if |x| <> |y| then RETURN(0) fi
23 >     if |x| = 0 then RETURN(1) fi
24 >     if |x| = 1 then RETURN(B[x[1],y[1]]) fi
25 >     RETURN(sum_(j in 1..|x|) (-1)^(j-1)*B(x[1],y[j])*cup(x[2..-1],y/y[j],B))
26 >   end cup
27 > (now we compute the double sum, to gain efficiency we do this grade wise)
28 > (note that there are r over NX r-vectors in psetx, analogously for psety)
29 > max_grade <- |lstx <- convert_to_set union lsty <- convert_to_set|
30 > res <- 0, pos1 <- 0
31 > for j from 0 to NX      (iterate over all j-vectors of psetx)
32 >   begin
33 >     F1 <- N1!/((N1-j)!*j!)  (number of terms (N1 over j))
34 >     pos2 <- 0
35 >     for i from 0 to min(N2,max_grade-j)
36 >       (iterate over all i-vectors of psety not exceeding max_grade while
37 >       others are zero)
38 >       begin
39 >         F2 <- N2!/((N2-i)!*i!)  (number of terms (N2 over i))
40 >         for n from 1 to F1      (for all j-vectors)
41 >           begin
42 >             for m from 1 to F2  (for all i-vectors)
43 >               begin
44 >                 res <- res +
45 >                 pSgnx[pos1+n]*pSgny[pos2+m]*
46 >                 *cup(fun1(psetx[PN1-pos1-n]),fun2(psety[pos2+m]),lname)*
47 >                 makeclibasmon -> ([fun1 -> psetx[pos1+n],fun2 -> psety[PN2-pos2-m]])]
48 >               end
49 >             end
50 >           pos2 <- pos2+F2
51 >           end
52 >         pos1 <- pos1+F1;
53 >       end
54 >     reorder -> res (reorder basis elements in res into standard order)
55 >   end cmulRS

```

---

It is clear from this algorithm that only those terms are considered which might

be non-zero: If all  $B_{i,j}$  are non-zero and different so that no cancellation takes place between them, all terms will survive. The combinatorial power of the Hopf algebraic approach is clearly demonstrated with this algorithm and its superior behavior shows up in benchmarks (Abłamowicz and Fauser, 2003a).

#### 4. Dotted and undotted Graßmann bases in quantum Clifford algebras

It was shown above that CLIFFORD uses the Graßmann algebra  $\bigwedge V$  as the underlying vector space of the Clifford algebra  $Cl(V, B)$ . Thus, the Graßmann wedge basis of monomials is the standard basis used in CLIFFORD. A general element  $u$  in  $Cl(V, B)$  can be therefore viewed as a Graßmann polynomial.

When the bilinear form  $B$  has an antisymmetric part  $F = -F^T$ , it is convenient to split  $B = g + F$ , where  $g$  is the symmetric part of  $B$ , and to introduce the so called “dotted Graßmann basis” (Abłamowicz and Lounesto, 1996) and the dotted wedge product  $\dot{\wedge}$ . The old Graßmann basis will be referred to here as “undotted Graßmann basis”. In CLIFFORD, the wedge product is given by the procedure `wedge` and  $\&w$  while the dotted wedge product is given by `dwedge` and  $\&dw$ .

According to the Chevalley definition of the Clifford product  $\&c$ , we have

$$\mathbf{x} \&c u = \mathbf{x} \lrcorner_B u + \mathbf{x} \&w u = \text{LC}(\mathbf{x}, u, B) + \text{wedge}(\mathbf{x}, u) \quad (4)$$

for a 1-vector  $\mathbf{x}$  and an arbitrary element  $u$  of  $Cl(B)$ . Here,  $\text{LC}(\mathbf{x}, u, B)$  denotes the left contraction of  $u$  by  $\mathbf{x}$  with respect to the entire bilinear form  $B$ . However, when  $B = g + F$  then

$$\mathbf{x} \lrcorner_B u = \text{LC}(\mathbf{x}, u, B) = \mathbf{x} \lrcorner_g u + \mathbf{x} \lrcorner_F u = \text{LC}(\mathbf{x}, u, g) + \text{LC}(\mathbf{x}, u, F) \quad (5)$$

and

$$\mathbf{x} \&c u = \text{LC}(\mathbf{x}, u, B) + \mathbf{x} \&w u \quad (6)$$

$$= \text{LC}(\mathbf{x}, u, g) + \text{LC}(\mathbf{x}, u, F) + \mathbf{x} \&w u \quad (7)$$

$$= \text{LC}(\mathbf{x}, u, g) + \mathbf{x} \&dw u \quad (8)$$

where  $\mathbf{x} \&dw u - \mathbf{x} \&w u = \text{LC}(\mathbf{x}, u, F)$ . That is, the wedge and the dotted wedge “differ” by the contraction term(s) with respect to the antisymmetric part  $F$  of  $B$ . This dotted wedge  $\&dw$  can be extended to elements of higher grades. Its properties are discussed next.

##### 4.1. Indexing `dwedge` and $\&dw$

Procedure `dwedge` (and its infix form  $\&dw$ ) requires an index which can be a symbol or an antisymmetric matrix. That is, `dwedge` computes the dotted wedge product of two Graßmann polynomials and expresses its answer in the undotted

basis. Special procedures exist which convert polynomials between the undotted and dotted bases. When no index is used, the default is  $F$  :

```
> dwedge[K] (e1+2*e2we3, e4+3*e1we2); &dw(ei+2*ejwek, ei+2*ejwek);
-(-K1,4 + 6 K2,3 K1,2) Id - 6 K1,2 e2we3 - 6 K2,3 e1we2 - 2 K2,4 e3 + 2 K3,4 e2
- 3 K1,2 e1 + e1we4 + 2 e2we3we4
4 eiwejwek - 4 Fi,k ej + 4 Fi,j ek - 8 Fj,k ejwek - 4 Fj,k^2 Id
```

Observe that conversion from the undotted wedge basis to the dotted wedge basis using antisymmetric form  $F$  and `dwedge[F]` are related through the following `convert` function:

$$\text{dwedge}[F](e_1, e_2, \dots, e_n) = \text{convert}(e_1we_2w\dots we_n, \text{wedge\_to\_dwedge}, F)$$

which can be shown as follows:

```
> F:=array(1..9,1..9,antisymmetric);
> dwedge[F] (e1,e2)=convert(wedge(e1,e2),wedge_to_dwedge,F);
e1we2 + F1,2 Id = e1we2 + F1,2 Id
> dwedge[F] (e1,e2,e3)=convert(wedge(e1,e2,e3),wedge_to_dwedge,F);
e1we2we3 + F2,3 e1 - F1,3 e2 + F1,2 e3 = e1we2we3 + F2,3 e1 - F1,3 e2 + F1,2 e3
```

## 4.2. Associativity of dwedge

Operation `dwedge` is associative with the unity  $\mathbf{1} = \text{Id}$  as a unit:

```
> evalb(dwedge[F] (dwedge[F] (e1,e2),e3)=dwedge[F] (e1,dwedge[F] (e2,e3)));
true
```

For some arbitrary random Clifford polynomials<sup>††</sup>  $u, v, z$  expressed in Graßmann undotted basis we can show associativity as follows:

```
> u:=2*Id+e1-3*e2we3:v:=3*Id-4*e1we3+e7:z:=4*Id-2*e3+e1we2we3;
> evalb(dwedge[F] (Id,u)=u),evalb(dwedge[F] (u,Id)=u);
true, true
> evalb(dwedge[F] (dwedge[F] (u,v),z)=dwedge[F] (u,dwedge[F] (v,z)));
true
```

We have, therefore, the following identity that is satisfied by any two elements  $u$  and  $v$  in  $Cl(B)$ ,  $B = g + F$ , that are, by default, expressed in terms of the undotted Graßmann basis:

$$u \wedge v = (u_F \hat{\wedge} v_F)_{-F}. \quad (9)$$

Here  $u_F$  and  $v_F$  are the elements  $u$  and  $v$  expressed in the dotted basis with respect to the form  $F$  while  $(\dots)_{-F}$  denotes conversion back from the dotted basis to the undotted basis w.r.t.  $-F = F^T$ . This can be illustrated in CLIFFORD as follows:

<sup>††</sup>In CLIFFORD ver. 6 and higher there are three procedures useful for testing that return a random Graßmann basis monomial, a random monomial and a random polynomial, respectively. See `?rd_clibasmon`, `?rd_climon`, `?rd_clipolynom`.

```

> uu:=convert(u,wedge_to_dwedge,F); #u converted to dotted basis
vv:=convert(v,wedge_to_dwedge,F); #v converted to dotted basis
      uu := e1 - 3 e2we3 - 3 F2,3 Id + 2 Id
      vv := 3 Id - 4 e1we3 - 4 F1,3 Id + e7
> out1:=dwedge[F](uu,vv): #dwedge computed w.r.t. F
> out2:=convert(out1,dwedge_to_wedge,-F); #back to undotted basis
      out2 := 3 e1 - 9 e2we3 + 6 Id - 8 e1we3 + e1we7 - 3 e2we3we7 + 2 e7
> out3:=wedge(u,v); #direct computation of wedge product
      out3 := 3 e1 - 9 e2we3 + 6 Id - 8 e1we3 + e1we7 - 3 e2we3we7 + 2 e7
and it can be seen that out2 = out3.

```

### 4.3. Dotted and undotted wedge bases

The default Graßmann basis in  $Cl(B)$  used in CLIFFORD is undotted. However, one can easily use the dotted basis. For example, we expand the basis of the original wedge into the dotted wedge and back. For this purpose we choose  $\dim(V) = 3$  and consider  $Cl(B)$  with the antisymmetric part  $F$ . The undotted wedge basis for  $\bigwedge V$  is then:

```

> w_bas:=cbasis(dim_V); ## the wedge basis
      w_bas := [Id, e1, e2, e3, e1we2, e1we3, e2we3, e1we2we3]

```

Now we map the convert function onto this basis to get the dotted wedge basis:

```

> d_bas:=map(convert,w_bas,wedge_to_dwedge,F);
test_wbas:=map(convert,d_bas,dwedge_to_wedge,-F);
      d_bas := [Id, e1, e2, e3, e1we2 + F1,2 Id, e1we3 + F1,3 Id, e2we3 + F2,3 Id,
e1we2we3 + F2,3 e1 - F1,3 e2 + F1,2 e3]
      test_wbas := [Id, e1, e2, e3, e1we2, e1we3, e2we3, e1we2we3]

```

Notice that only the unity  $\mathbf{1}$  and the one vector basis elements  $\mathbf{e}_i$  remain unaltered and that the other basis elements of higher grades pick up additional terms of lower grades (which preserves the filtration). It is possible to define aliases in CLIFFORD for the dotted wedge basis “monomials” similar to the Graßmann basis monomials. For example, we could denote the element  $e1we2 + F[1,2] * Id$  by  $e1We2 (= \mathbf{e}_1 \wedge \mathbf{e}_2)$  and similarly for other elements:

```

> alias(e1We2=e1we2 + F[1,2]*Id,e1We3=e1we3 + F[1,3]*Id,
e2We3=e2we3 + F[2,3]*Id,
e1We2We3=e1we2we3+F[2,3]*e1-F[1,3]*e2+F[1,2]*e3);
      I, e1We2, e1We3, e2We3, e1We2We3

```

and then Maple will display automatically dotted basis in terms of the aliases:

```

> d_bas;
      [Id, e1, e2, e3, e1We2, e1We3, e2We3, e1We2We3]

```

That is, as linear spaces we find isomorphisms

$$\begin{aligned}
Cl(B) &\cong \langle \mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \rangle \\
&\cong \langle \mathbf{1}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \rangle
\end{aligned}$$

where  $\mathbf{e}_1 \wedge \mathbf{e}_2 = e1We2$ , etc.

#### 4.4. Contraction in dotted and undotted bases

The contraction  $\lrcorner_K$  w.r.t. any bilinear form  $K$  works on both undotted and dotted bases in the same manner which can be seen if we re-convert the dotted basis after the computation into the wedge (undotted) basis. In a reasonable setting, the antisymmetric bilinear form  $F$  would be the antisymmetric part of  $B$ . To read more about the left contraction  $\text{LC}$  in  $\mathcal{Cl}(B)$  check the help page for  $\text{LC}$  or see (Ablamowicz and Lounesto, 1996). We have the following identity for any two elements  $u$  and  $v$  in  $\mathcal{Cl}(B)$  expressed in the undotted Graßmann basis:

$$v \lrcorner_B u = (v \lrcorner_B u_F)_{-F} \quad (10)$$

As before,  $u_F$  is the element  $u$  expressed in the dotted basis while  $(\dots)_{-F}$  accomplishes conversion back to the undotted basis. To illustrate this fact, we first contract from the left an arbitrary element  $u$  in  $\mathcal{Cl}(B)$  by  $\mathbf{1}, \mathbf{e}_i, \mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$  (here we limit our example to  $\dim(V) = 3, 1 \leq i, j, k \leq 3$ ) and then we extend it to a left contraction by an arbitrary element  $v$  in  $\mathcal{Cl}(B)$ .

```
> u:=add(x.i*w_bas[i+1],i=0..7):uF:=convert(uw,wedge_to_dwedge,F):
  v:=add(y.i*w_bas[i+1],i=0..7):
```

Contraction with respect to  $\mathbf{1}$  :

```
> evalb(LC(Id,u,B)=convert(LC(Id,uF,B),dwedge_to_wedge,-F));
      true
```

Contraction with respect to  $\mathbf{e}_i$  :

```
> evalb(LC(ei,u,B)=convert(LC(ei,uF,B),dwedge_to_wedge,-F));
      true
```

Contraction with respect to  $\mathbf{e}_i \wedge \mathbf{e}_j$  :

```
> evalb(LC(eiwej,u,B)=convert(LC(eiwej,uF,B),dwedge_to_wedge,-F));
      true
```

Contraction with respect to  $\mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k$  :

```
> evalb(LC(eiwejwkek,u,B)=convert(LC(eiwejwkek,uF,B),dwedge_to_wedge,-F));
      true
```

Finally, contraction with respect to an arbitrary element  $v$  :

```
> evalb(LC(vw,uw,B)=convert(LC(vw,uF,B),dwedge_to_wedge,-F));
      true
```

#### 4.5. Clifford product in dotted and undotted bases

We can build a Clifford algebra  $\mathcal{Cl}(B)$  over each basis set, that is, over the undotted or dotted Graßmann basis, but with different bilinear forms:  $B = g$  or  $B = g + F$  respectively (following notation from (Ablamowicz and Lounesto, 1996)). Let's compute various Clifford products with respect to the symmetric form  $g$  and with respect to the full form  $B$  using procedure `cmul` that takes a bilinear form as its index. As an example, we will use two most general elements  $u$  and  $v$  in  $\bigwedge V$  when  $\dim(V) = 3$ . Most output will be eliminated.

```
> u:=add(x.k*w_bas[k+1],k=0..7):v:=add(y.k*w_bas[k+1],k=0..7):
```

We can then define in  $\bigwedge V$  Clifford product `cmul[g]` with respect to the symmetric part  $g$  and another Clifford product `cmul[B]` with respect to the entire form  $B$ :

```
> cmulg:=proc() RETURN(cmul[g](args)) end:
  cmulB:=proc() RETURN(cmul[B](args)) end:
```

Thus, we are ready to perform computations around our next commutative diagram, however output will be eliminated to save space. First, we compute Clifford product `cmul[g](u,v)` in  $\mathcal{Cl}(g)$  in undotted Graßmann basis.

```
> uv:=cmulg(u,v): ## Clifford product w.r.t. g in Cl(g) in wedge basis
```

Now, we convert  $u$  and  $v$  to  $u_F$  and  $v_F$ , respectively, expressed in the dotted wedge basis:

```
> uF:=convert(u,wedge_to_dwedge,F):vF:=convert(v,wedge_to_dwedge,F):
```

We now compute the Clifford product of  $u_F$  and  $v_F$  in  $\mathcal{Cl}(B)$  in the dotted wedge basis,

```
> uFvF:=cmulB(uF,vF): ## Clifford product in Cl(B) in dwedge basis
```

convert back the above result back to the undotted wedge basis:

```
> uv2:=convert(uFvF,dwedge_to_wedge,-F): ## convert result dwedge->wedge
```

and verify that the results are the same:

```
> simplify(uv-uv2): ## show equality !
```

0

Thus, we have shown that the following identity involving `cmul[g]` and `cmul[B]` is true (at least when  $\dim(V) = 3$ ):<sup>‡‡</sup>

$$(uv)_g = u \&_{\mathcal{C}_g} v = (u_F \&_{\mathcal{C}_B} v_F)_{-F} = ((u_F v_F)_B)_{-F} \quad (11)$$

This shows that the Clifford algebra  $\mathcal{Cl}(g)$  of the symmetric part  $g$  of  $B$  using the undotted exterior basis is isomorphic, as an associative algebra, to the Clifford algebra  $\mathcal{Cl}(B)$  of the entire bilinear form  $B = g + F$  spanned by the dotted wedge basis if the antisymmetric part  $F$  of  $B$  is exactly the same as  $F$  is used to connect the two basis (Ablamowicz and Lounesto, 1996).

#### 4.6. Reversion in dotted and undotted bases

We proceed to show that the expansion of the Clifford basis elements into the dotted or undotted exterior products has also implications for other well known operations such as the Clifford reversion anti-automorphism  $\tilde{\cdot} : \mathcal{Cl}(B) \rightarrow \mathcal{Cl}(B)$ ,  $uv \mapsto \tilde{v}\tilde{u}$ , which preserves the grades in  $\bigwedge V$  [but not in  $\bigwedge V$  unless  $B$  is symmetric.] Only when the bilinear form is symmetric, we find that the reversion is grade preserving, otherwise it reflects only the filtration: That is, reversed elements are in general sums of terms of the same and lower degrees.

```
> reversion(e1we2,B): #reversion with respect to B
  reversion(e1we2,g): #reversion with respect to g (classical result)
```

$$-e1we2 - 2F_{1,2} Id$$

<sup>‡‡</sup>Here,  $(uv)_g$  is the Clifford product with respect to  $g$  while  $u_F \&_{\mathcal{C}_B} v_F$  and  $(u_F v_F)_B$  are the Clifford products with respect to  $B$ , that is, in  $\mathcal{Cl}(g)$  and  $\mathcal{Cl}(B)$ , respectively.

$$-e1we2$$

Observe in the above that only when  $B_{1,2} = B_{2,1}$ , the result  $-\mathbf{e}_1 \wedge \mathbf{e}_2$  known from the theory of classical Clifford algebras is obtained. Likewise,

```
> cbas:=cbasis(3);
```

$$cbas := [Id, e1, e2, e3, e1we2, e1we3, e2we3, e1we2we3]$$

```
> map(reversion,cbas,B);
```

$$[Id, e1, e2, e3, -e1we2 - 2F_{1,2} Id, -e1we3 - 2F_{1,3} Id, -e2we3 - 2F_{2,3} Id, \\ -2F_{2,3} e1 + 2F_{1,3} e2 - 2F_{1,2} e3 - e1we2we3]$$

If instead of  $B$  we use a symmetric matrix  $g = g^T$  (or the symmetric part of  $B$ ), then

```
> map(reversion,cbas,g);
```

$$[Id, e1, e2, e3, -e1we2, -e1we3, -e2we3, -e1we2we3]$$

Convert now  $\mathbf{e}_1 \wedge \mathbf{e}_2$  to the dotted basis to get  $\mathbf{e}_1 \dot{\wedge} \mathbf{e}_2 = e1We2$ :

```
> convert(e1we2,wedge_to_dwedge,F);
```

$$e1We2$$

Apply reversion to  $e1We2$  with respect to  $F$  to get the reversed element in the dotted basis:

```
> reversed_e1We2:=reversion(e1We2,F);
```

$$reversed_e1We2 := -e1we2 - F_{1,2} Id$$

Observe, that the above element is equal to the negative of  $e1We2$  just like reversing  $e1we2$  with respect to the symmetric part  $g$  of  $B$ :

```
> reversed_e1We2+e1We2;
```

$$0$$

Finally, convert reversed  $e1We2$  to the undotted standard Graßmann basis to get  $-e1we2$ :

```
> convert(reversed_e1We2,dwedge_to_wedge,-F);
```

$$-e1we2$$

The above, of course, can be obtained by applying reversion to  $e1we2$  with respect to the symmetric part  $g$  of  $B$ :

```
> reversion(e1we2,g); #reversion w.r.t. the symmetric part g
```

$$-e1we2$$

This shows that the dotted wedge basis is the particular basis which is stable under the Clifford reversion computed with respect to  $F$ , the antisymmetric part of the bilinear form  $B$ . This requirement allows one to distinguish Clifford algebras  $\mathcal{Cl}(g)$  which have a symmetric bilinear form  $g$  from those which do not have such symmetric bilinear form but a more general form  $B$  instead. We call the former **classical Clifford algebras** while we use the term **quantum Clifford algebras** for the general not necessarily symmetric case (Ablamowicz and Fauser, 2000).

## 5. Conclusions

This paper continues with the second part (Ablamowicz and Fauser, 2003b) about BIGEBRA where further aims and outlooks for the future applications of CLIFFORD and BIGEBRA are given.

Appendix A: Code of `cmulNUM`

Here is a shortened code of the recursive procedure `cmulNUM`.

---

```

56 > cmulNUM(a1,a2,B) [a1, a2 - two Grassmann monomials, B - name of bilinear form]
57 > begin
58 >   if nargs<>3 then ERROR('exactly three arguments are needed') fi
59 >   if has(0,map(simplify,[a1,a2])) then RETURN(0) fi
60 >   if a2='Id' then RETURN(a1) fi
61 >   if a1='Id' then RETURN(a2) fi
62 >   L <- indices from a1
63 >   N <- length of L
64 >   coB,nameB <- coefficient of B, B [to handle -B]
65 >   if N=0 then RETURN(coeff(a1,Id)*a2) elif N=1 then
66 >     L2 <- list of indices from a2
67 >     RETURN(reorder(simplify(makeclibasmon([L[1],op(L2)]))
68 >       +add((-1)^(i-1)*coB*nameB[L[1],L2[i]]*
69 >         makeclibasmon(subs(L2[i]=NULL,L2)),i=1..nops(L2))))
70 >   elif N=2 then
71 >     x1 <- substring(a1,1..2)
72 >     x2 <- substring(a1,4..5)
73 >     p2 <- procname(x2,a2,B)
74 >     S <- clibilinear(x1,p2, procname,B)
75 >     RETURN(simplify(S-coB*nameB[op(L)]*a2))
76 >   fi
77 >   x <-cat(e,L[-1])
78 >   p1<-substring(a1,1..(3*N-4))
79 >   p2<- procname(x,a2,B)
80 >   S<-clibilinear(p1,p2, procname,B)
81 >     -add((-1)^(i)*coB*nameB[L[-i],L[-1]]*
82 >       procname(makeclibasmon(subs(L[-i]=NULL,L[1..-2])),a2,B),i=2..N)
83 >   RETURN(reorder(simplify(S)))
84 > end cmulNUM

```

---

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