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# TABLES OF INTEGRALS, THE COSECANT FUNCTION AND CONSISTENCY

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**TABLES OF INTEGRALS, THE COSECANT FUNCTION, AND CONSISTENCY**

When I was a graduate student, the first two calculus books I taught out of ([1], [2]) wrote the integrals of the six trigonometric functions in the form:

$$\int \sin u du = -\cos u + C \tag{1}$$

$$\int \cos u du = \sin u + C \tag{2}$$

$$\int \tan u du = -\ln |\cos u| + C \tag{3}$$

$$\int \cot u du = \ln |\sin u| + C \tag{4}$$

$$\int \sec u du = \ln |\sec u + \tan u| + C \tag{5}$$

$$\int \csc u du = -\ln |\csc u + \cot u| + C. \tag{6}$$

The integrals (1)-(6), written in this manner, display a delightful mnemonic duality of cofunctions and minus signs. Nevertheless, this pleasant pattern is lost on most authors of current calculus textbooks ([3], [4], ... , [$n$]), who instead focus on the fact that equations (3) and (6) contain that bane of mathematicians everywhere: an avoidable initial minus sign. For this reason, (3) and (6) are now usually replaced by the equivalent forms:

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$$\int \tan u du = \ln|\sec u| + C \tag{3'}$$

and

$$\int \csc u du = \ln|\csc u - \cot u| + C. \tag{6'}$$

Unfortunately, the authors of the books that use these latter forms have created inconsistencies in their tables of integrals, because they have neglected the fact that the various integration formulas are not independent, but interdependent. The integral formulas obtained through the method of trigonometric substitution naturally depend on the integrals of the trig functions, and if the form of the latter is altered, the form of the former should reflect that change, but in the integral tables in current calculus books, this consistency feature is not exhibited.

By working out the integrals for which trig substitution is appropriate (and using the sine, tangent, or secant functions for the substitution instead of the cofunctions, as is now standard procedure) it can be shown that the only ones which call for the integral of the tangent function are ones that can be done using more direct methods, such as partial fractions. However, there are several that call for the integral of the cosecant function and the formulas for these integrals should reflect the form of the cosecant function integral being used. Specifically, if the formula is as in (6), then the integral formulas in the left column, equations (7) – (18), should be listed, and if the form used is as in (6′), then the integral formulas in the right column, equations (7′) – (18′), should be listed.

$$\int \csc u du = -\ln|\csc u + \cot u| + C \tag{6}$$

$$\int \frac{\sqrt{a^2+u^2}}{u} du = \sqrt{a^2+u^2} - a\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C \tag{7}$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C \tag{8}$$

$$\int \frac{\sqrt{a^2-u^2}}{u} du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{9}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{10}$$

$$\int \frac{\sqrt{a^2+u^2}}{u^3} du = -\frac{\sqrt{a^2+u^2}}{2u^2} - \frac{1}{2a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C \tag{11}$$

$$\int \frac{du}{u^3\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{2a^2u^2} + \frac{1}{2a^3}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C \tag{12}$$

$$\int \frac{\sqrt{a^2-u^2}}{u^3} du = -\frac{\sqrt{a^2-u^2}}{2u^2} + \frac{1}{2a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{13}$$

$$\int \frac{du}{u^3\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{2a^2u^2} - \frac{1}{2a^3}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{14}$$

$$\int \frac{du}{u\sqrt{(a^2+u^2)^3}} = \frac{1}{a^2\sqrt{a^2+u^2}} - \frac{1}{a^3}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C \tag{15}$$

$$\begin{aligned}\int \frac{du}{u^3\sqrt{(a^2+u^2)^3}} = -\frac{1}{2a^2u^2\sqrt{a^2+u^2}} - \frac{3}{2a^4\sqrt{a^2+u^2}} \\ + \frac{3}{2a^5}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C\end{aligned} \tag{16}$$

$$\int \frac{du}{u\sqrt{(a^2-u^2)^3}} = \frac{1}{a^2\sqrt{a^2-u^2}} - \frac{1}{a^3}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C \tag{17}$$

$$\begin{aligned}\int \frac{du}{u^3\sqrt{(a^2-u^2)^3}} = -\frac{1}{2a^2u^2\sqrt{a^2-u^2}} + \frac{3}{2a^4\sqrt{a^2-u^2}} \\ - \frac{3}{2a^5}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C\end{aligned} \tag{18}$$

$$\int \csc u du = \ln|\csc u - \cot u| + C \tag{6'}$$

$$\int \frac{\sqrt{a^2+u^2}}{u} du = \sqrt{a^2+u^2} + a \ln\left|\frac{\sqrt{a^2+u^2}-a}{u}\right| + C \tag{7'}$$

$$\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{1}{a} \ln\left|\frac{\sqrt{a^2+u^2}-a}{u}\right| + C \tag{8'}$$

$$\int \frac{\sqrt{a^2-u^2}}{u} du = \sqrt{a^2-u^2} + a \ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C \tag{9'}$$

$$\int \frac{du}{u\sqrt{a^2-u^2}} = \frac{1}{a} \ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C \tag{10'}$$

$$\int \frac{\sqrt{u^2+a^2}}{u^3} du = -\frac{\sqrt{u^2+a^2}}{2u^2} + \frac{1}{2a} \ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C \tag{11'}$$

$$\int \frac{du}{u^3\sqrt{u^2+a^2}} = -\frac{\sqrt{u^2-a^2}}{2a^2u^2} - \frac{1}{2a^3} \ln\left|\frac{\sqrt{u^2+a^2}-a}{u}\right| + C \tag{12'}$$

$$\int \frac{\sqrt{a^2-u^2}}{u^3} du = -\frac{\sqrt{a^2-u^2}}{2u^2} - \frac{1}{2a} \ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C \tag{13'}$$

$$\int \frac{du}{u^3\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{2a^2u^2} + \frac{1}{2a^3} \ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C \tag{14'}$$

$$\int \frac{du}{u\sqrt{(a^2+u^2)^3}} = \frac{1}{a^2\sqrt{a^2+u^2}} + \frac{1}{a^3}\ln\left|\frac{\sqrt{a^2+u^2}-a}{u}\right| + C \tag{15'}$$

$$\begin{aligned}\int \frac{du}{u^3\sqrt{(a^2+u^2)^3}} = -\frac{1}{2a^2u^2\sqrt{a^2+u^2}} &- \frac{3}{2a^4\sqrt{a^2+u^2}} \\ &- \frac{3}{2a^5}\ln\left|\frac{\sqrt{a^2+u^2}-a}{u}\right| + C\end{aligned} \tag{16'}$$

$$\int \frac{du}{u\sqrt{(a^2-u^2)^3}} = \frac{1}{a^2\sqrt{a^2-u^2}} + \frac{1}{a^3}\ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C \tag{17'}$$

$$\begin{aligned}\int \frac{du}{u^3\sqrt{(a^2-u^2)^3}} = -\frac{1}{2a^2u^2\sqrt{a^2-u^2}} &+ \frac{3}{2a^4\sqrt{a^2-u^2}} \\ &+ \frac{3}{2a^5}\ln\left|\frac{a-\sqrt{a^2-u^2}}{u}\right| + C\end{aligned} \tag{18'}$$

Two comments can be made:

1. In both columns, all of the integrals are written so that inside the logarithm the term corresponding to the cosecant is listed first and the term corresponding to the cotangent is listed second. This seems only logical, and will guarantee that for the integrals in the right column, the numerator inside the logarithm will never be negative. This listing of terms in correct order should also be done for integrals where a trig substitution calls for the integral of the secant function, provided integrals involving $u^2 + a^2$ terms and integrals involving $u^2 - a^2$ terms are listed separately.
2. If the integrals in the second column are used in an integral table, there would result greater insight as to how the formulas are derived; specifically, if the formula has a difference of two terms inside a logarithm, it must have resulted from integrating a cosecant term after a trig substitution, and if the formula has a sum inside a logarithm, it must have resulted from integrating a secant term.

It may be time to take a new look at tables of integrals. In this day of increasingly sophisticated technology and advances in integration by computer, one might wonder if such matters merit consideration. In response, it must be mentioned that use of technology should not come at the expense of consistency, or a view of the big picture. Besides, no matter what happens with computers, there will still be, at least for the forseeable future, calculus textbooks with integral tables. A fresh approach with an eye toward consistency, symmetry, and instructional value, rather than a mere copying from other books or old editions, would be most welcome.

**Postscript.** For similar reasons, if the sine, tangent, and secant functions are to be used in trigonometric substitutions, there is no reason to use the inverse cosine function in integrals containing a $2au - u^2$ term. In each case, the $\cos^{-1}(1 - \frac{u}{a})$ should be replaced by $\sin^{-1}(\frac{u}{a} - 1)$.

## REFERENCES


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