VARIOUS ESTIMATIONS OF π AS DEMONSTRATIONS OF THE MONTE CARLO METHOD
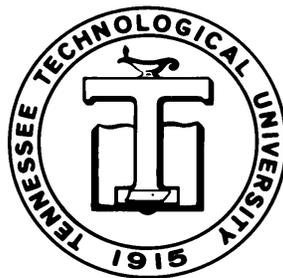
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1. Introduction

The Monte Carlo method uses pseudo-random numbers (numbers which are generated by a formula using the selection of one random "seed" number) as values for certain variables in algorithms to generate random variates of chosen probability functions to be used in simulations of statistical models or to generate pseudo-random numbers for more complex probability functions . In this paper we will demonstrate a unique use of the Monte Carlo method by using it in several procedures to estimate the value of π.

2. History of the search for π

The first theoretical calculations of π appear to have been made by the scholar Archimedes (287-212 BC). Archimedes calculated π to be between 223/71 and 22/7. He used the system of a circle with inscribed and ascribed polygons. Letting the number of sides of the polygons increase until he had polygons with 96 sides. Archimedes noted that as he increased the number of sides, the semiperimeter of the inscribed polygon formed an increasing sequence, while the semiperimeter of the ascribed polygon formed a decreasing sequence. He further noted that both sequences then had a limit of π.

The next significant work on π was not accomplished until 400 years later, around 150 AD by Ptolemy. Ptolemy was able to calculate π to 3.1416. From Ptolemy to Van Ceulen in the early 15th century, who calculated π to 35 decimal places, there was little improvement in Archimedes technique, only increased stamina in carrying out the calculations. It was during the latter portion of the 15th century that renewed interest in the calculation of π brought about improved mathematical techniques by mathematicians such

as Wallis, James Gregory and Leibniz. Another major step came in 1761 when Lambert was able to prove that π was irrational.

In the past 50 years, it has been a yardstick of any new super-computer to determine π to increasingly higher decimal places. Mathematicians are still searching for better and better methods to estimate π.

Note: Most of the history detailed here comes from O'Connor and Robertson (1996)

3. Methods

We investigated several methods of estimating π before deciding which ones would work well with the Monte Carlo method. The first method chosen was Buffons Needle Method. This method uses a group of needles dropped at random onto a surface with equally spaced parallel lines. The space between the lines must be at least the length of a needle. π is then estimated with the formula:

$$\pi \approx 2 \times \frac{\text{Length of the needle}}{\text{Distance between the lines}} \times \frac{\text{Number of needle tosses}}{\text{Number of times needle crosses a line}}$$

We will not give a formal proof here, but this method is outlined in detail by Schroeder (1974). This is a very good example of using the Monte Carlo method to estimate π. This Monte Carlo simulation was accomplished using a software packages written specifically to do Buffons Needle Method. The version we used was obtained from www.efg2.com/lab. Our results are displayed in Table 3.1. We conducted thirty simulations (B=30) for each value of the number of needle drops and estimated π as the mean of the thirty values obtained. The accuracy is the standard deviation of the obtained mean.

Table 3.1

Needle drops	100	1000	10000	100000	1000000	10000000
Estimate of π	3.2	3.1	3.14	3.139	3.143	3.1421
Accuracy	0.3	0.9	0.02	0.009	0.002	0.0008

The second method that we chose involved a system of ratios of the area of a circle to the area of a square. We liked this method because it seemed well suited to the Monte Carlo method. A random selection of numbers was obtained from the uniform distribution from q to q. Thus, we produced a Cartesian plot with $x \sim U(-q, q)$ and $y \sim U(-q, q)$. We then calculated all

the points that would fall within a radius of q from the center of the square and thus formed an inscribed circle. This is shown in Figure 3.1.
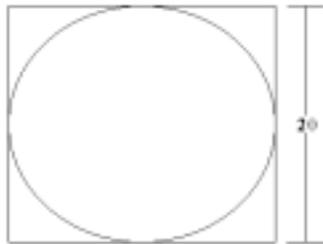

Figure 3.1

Next, if we take a ratio of the area of the circle to the area of the square we get:

$$A_c/A_s = \frac{\pi\theta^2}{4\theta^2}$$

$$\Rightarrow \pi = 4(A_c/A_s)$$

We approximate the area of the square and the circle by simply counting the number of points that fall within each one. Therefore, our approximation of π is given by:

$\pi \approx 4(\text{Number of points in the circle/Number of points in the square})$

The obtained results are displayed in Table 3.2. We again used B=30.

Table 3.2

Total points	100	1000	10000	100000	1000000	10000000
Estimate of π	3.14	3.144	3.1359	3.14160	3.141690	3.1416010
Accuracy	0.01	0.003	0.0001	3.0×10^{-5}	5.0×10^{-6}	2.0×10^{-7}

This last method appears to be the best out of the three that we will demonstrate. A significant factor in the accuracy was the quality of the

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random number generator in the software package. The better the package was in generating a random scatter of points, the more accurate our estimate was.

Our final method involved the use of Importance Sampling Algorithms. As detailed by Cheng (2000), these algorithms use random sampling from a specified to distribution to approximate an integral. Take the integral

$$\int g(x)dx$$

This integral can be equivalently expressed as

$$\int \frac{g(x)}{f(x)}f(x)dx$$

where $f(x)$ is the importance function. The importance function is a probability density function over the integral range Ω where some algorithm exists for generating samples from $f(x)$ and where if $g(x) \neq 0$ then $f(x) \neq 0$. After sampling n points, $s_1, s_2, \ldots, s_n$, from Ω according to the p.d.f. $f(x)$, we can estimate the integral by

$$\widetilde{I_n} \approx \frac{1}{n}\sum_{i=1}^{n}\frac{g(s_i)}{f(s_i)}$$

If one chooses $f(x)$ to be the the p.d.f. of the uniform distribution, as we did, then importance sampling becomes a general Monte Carlo sampling method. For our $g(x)$ we chose a function based on the probability distribution function of the normal distribution. That is:

$$h(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

Then the p.d.f. for a standard normal (i.e., $\mu = 0$ and $\sigma^2 = 1$) is given by:

$$h(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty$$

We know that the integral of this function must equal 1, so it is easy to see that the integral of the function:

$$g(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}$$

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Therefore, in order to estimate π using this procedure we simply to draw a sample of sample size n from the uniform distribution and use these values in our Importance Sampling Algorithm. This gives us a value:

$$\widetilde{I_n} \approx \sqrt{2\pi}$$

Therefore, we simply solve the value of our algorithm for π. This then gives us:

$$\pi \approx \frac{\left(\widetilde{I_n}\right)^2}{2}$$

The results from these simulations are displayed in Table 3.3.

Table 3.3

Value of n	100	1000	10000	100000	1000000	10000000
Estimate of π	4.0	3.5	3.15	3.143	3.1398	3.14206
Accuracy	2.0	0.2	0.02	0.002	0.0002	2.0×10^{-5}

Unfortunately, the accuracy of this method was terrible and no definite conclusion could be made at this time. The author is pursuing further research into the cause of this error.

4. Conclusions

For this project, we attempted to demonstrate the Monte Carlo Method by using it in several procedures for estimating π. We have shown that there are several techniques, which yield a good estimate of π. We have also seen some decent methods of estimating π, which do not work well with the Monte Carlo Method. We have tried to show the reader how the Monte Carlo Method can be used in estimating figures in several problems.

5. References

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