ON OPTIMAL ROW-COLUMN DESIGNS FOR TWO TREATMENTS

David Caughran
Nizam Uddin
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On optimal row-column designs for two treatments

David Caughran and Nizam Uddin
Mathematics Department
Tennessee Technological University
Cookeville, TN 38505, USA

Abstract: This paper presents optimal $3 \times 3$, $3 \times 4$, and $3 \times 5$ row-column designs for two treatments in the presence of a symmetric doubly geometric correlated errors. All designs presented here minimize the variance of generalized least squares estimator of the difference between treatment effects under a correlated model that incorporates both fixed row and fixed column effects.

1. Introduction

An allocation of of $v \geq 2$ treatments to $pq$ experimental units, which are further grouped into $p$ rows and $q$ columns, is called a row-column design. This paper addresses the problem of choosing a row-column design $d$ that estimates the difference between the effects of two treatments with minimum variance under the model

$$Y_d = 1_{pq}\mu + Z_1\rho + Z_2\gamma + X_d\tau + \epsilon, \quad \text{cov}(\epsilon) = V. \quad (1.1)$$

Here $Y_d$ (written in column order) is the $pq \times 1$ response vector, $1_n$ is the $n \times 1$ column vector of ones, $\tau$ is the $v \times 1$ vector of treatment effects, $X_d$ is a $pq \times v$ plot-treatment design matrix that defines the allocation of treatments to the experimental units according to the design $d$, and $\rho$ and $\gamma$ are vectors of parameters for fixed row and fixed column effects, respectively. The matrices $Z_1 = 1_q \otimes I_p$ and $Z_2 = I_q \otimes 1_p$ are called the plot-row and plot-column incidence matrices, respectively. The error co-variance matrix is assumed here to be a special case ($\alpha = \beta$) of the following doubly geometric process:

$$V = \frac{\sigma^2(1-\alpha^2)^{-1}}{1-\beta^2} \left( \begin{array}{cccc} 1 & \beta & \beta^2 & \ldots & \beta^{q-1} \\ \beta & 1 & \beta & \ldots & \beta^{q-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta^{q-1} & \beta^{q-2} & \beta^{q-3} & \ldots & 1 \end{array} \right) \otimes \left( \begin{array}{ccccc} 1 & \alpha & \alpha^2 & \ldots & \alpha^{p-1} \\ \alpha & 1 & \alpha & \ldots & \alpha^{p-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha^{p-1} & \alpha^{p-2} & \alpha^{p-3} & \ldots & 1 \end{array} \right).$$

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With $Z = (1_{pq} Z_2 Z_1)$, the generalized least squares information matrix $C_d$ for estimation of treatment contrasts under (1.1) can be written as

$$C_d = X_d'V^{-1}X_d - X_d'V^{-1}Z(Z'V^{-1}Z)^{-1}Z'V^{-1}X_d.$$  \hspace{1cm} (1.2)

The matrix $C_d$, for any connected design $d$, is nonnegative definite with rank $v−1$. For $v = 2$, $C_d$ is of order $2 \times 2$ and of the following form

$$C_d = \begin{pmatrix} c_{d11} & -c_{d11} \\ -c_{d11} & c_{d11} \end{pmatrix}$$

where $c_{d11}$ must be determined by simplifying the matrix $C_d$ given by (1.2). This simplified form of $C_d$ implies that the $\text{var}_{d}(\hat{\tau}_1 - \hat{\tau}_2) = c_{d11}^{-1}$ for a design $d$. If we let $D(2, p, q)$ denote the class of all connected $p \times q$ row-column designs for two treatments, then a design $d^* \in D(2, p, q)$ is optimal if $\text{var}_{d^*}(\hat{\tau}_1 - \hat{\tau}_2) = c_{d^{*}11}^{-1} \leq c_{d11}^{-1} = \text{var}_{d}(\hat{\tau}_1 - \hat{\tau}_2)$ for all $d \in D(2, p, q)$. This is equivalent to saying that the design $d^*$ is optimal if $c_{d^{*}11} \geq c_{d11}$ for all $d \in D(2, p, q)$.

Several researchers have addressed the problems of optimality of row-column designs for $v \geq 2$ when observations are correlated (e.g., Gill and Shukla, 1985; Kunert, 1988; Martin, 1986; Martin and Eccleston, 1993; Morgan and Uddin, 1991, 1998; Uddin and Morgan, 1991, 1997a, 1997b; Uddin, 1997). Optimal $p \times q$ row-column designs under the model (1.1), that incorporates both row and column effects, are almost nonexistent except for some very special cases. Uddin and Morgan (1997a) have attempted to determine universally optimal two-dimensional block designs for correlated observations with and without row/column effects in (1.1). Their paper gives some universally optimal $p \times 2$ row-column designs under (1.1) when $v = 2$. For $v \geq 3$, their universally optimal designs use $b \geq 2$ blocks each of which is a $p \times 2$ row-column design. For $v = 2$, Uddin (1997) gives universally optimal $p \times q$ designs for all even $p$ and $q$ when both $\alpha$ and $\beta$ are positive. More specifically, it is shown in Uddin (1997) that the design

$$d^* = \begin{pmatrix} 1 & 2 & 1 & 2 & \ldots & 1 & 2 \\ 2 & 1 & 2 & 1 & \ldots & 2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 1 & 2 & \ldots & 1 & 2 \\ 2 & 1 & 2 & 1 & \ldots & 2 & 1 \end{pmatrix}_{p \times q}$$

is universally optimal for correlated observations.
is universally optimal (in the sense of Kiefer, 1975) over $D(v = 2, p, q)$ for all $\alpha \geq 0, \beta \geq 0$ and for all even $p$ and $q$. The author is not aware of any other paper that gives optimal designs in the present set-up. It appears that even for the special case of $v = 2$, the optimality problem is only partially solved under the model (1.1). For example, optimal $p \times q$ designs are not known for all $\alpha, \beta \in (-1, 1)$ when at least one of $p \geq 3$ and $q \geq 3$ is odd. At this time, we do not know if optimal $p \times q$ designs for two treatments can be determined for all $p, q, \alpha$ and $\beta$ mentioned above; see the information matrix $C_d$ in Uddin (1997) and the algebraic complexity involved in the determination of such optimal designs. However, for small $p$ and $q$, the problem can be solved by enumerating all possible designs for a given $p$ and $q$. In this paper, we have determined optimal $3 \times 3$, $3 \times 4$, and $3 \times 5$ designs for two treatments under (1.1) with $\alpha = \beta$. We have enumerated all possible designs in each case and determined optimal designs by comparing $c_{d11}$ of all designs for a given $p$ and $q$. Our results are presented in the following section.

2. Optimal designs for $v = 2$.  

We have utilized MAPLE software to simplify the information matrix $C_d$ and obtained $c_{d11}$ for all possible $d \in D(v = 2, p, q)$ for each combination of $p$ and $q$ mentioned above. Note that two treatments can be assigned to $pq$ experimental units in $2^{pq}$ ways, each of these arrangement is a $p \times q$ design. However, not all of these designs are connected since $C_d$ is a zero matrix for some $d$. In our search of optimal designs, we have calculated $c_{d11}$ element of $C_d$ for each connected design $d$. The optimal design is one that maximizes $c_{d11}$ over $D(v = 2, p, q)$ for $\alpha \in (-1, 1)$. However, no single design is found that maximizes $c_{d11}$ over $D(2, p, q)$ for all $\alpha \in (-1, 1)$. The optimal design depends on the magnitude of $p$, $q$ and $\alpha$.

In the following subsections, we use the convention that two designs $d_1$ and $d_2$ are distinct if $d_1$ can not be obtained from $d_2$ by interchanging the two symbols 1 and 2 in $d_2$, or $d_1$ can not be obtained by rotating the design $d_2$. 

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2.1 Optimal $3 \times 3$ designs for $v = 2$.

In this case, $c_{d11}$ of all connected $3 \times 3$ designs are obtained using MAPLE software. We have found four distinct designs $d_1$, $d'_1$, $d_2$, and $d_3$ such that the $\max(c_{d11}, c'_{d11}, c_{d211}, c_{d311})$, for each $\alpha \in (-1,1)$, is greater than or equal to the $c_{d11}$ values of all other $3 \times 3$ designs. Thus the $c_{d11}$ values of these four designs may be compared to determine optimal designs.

For the purpose of determining the values of $\alpha$ and the corresponding optimal designs, we first list in Table 1 these four distinct designs and the $c_{d11}$ element of the corresponding $C_d$.

<table>
<thead>
<tr>
<th>Design</th>
<th>$c_{d11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1 = \begin{pmatrix} 2 &amp; 2 &amp; 1 \ 2 &amp; 2 &amp; 1 \ 1 &amp; 1 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{-16 \alpha^4 + 32 \alpha^3 - 32 \alpha + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$d'_1 = \begin{pmatrix} 1 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 1 \end{pmatrix}$</td>
<td>$\frac{-16 \alpha^4 + 32 \alpha^3 - 32 \alpha + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$d_2 = \begin{pmatrix} 2 &amp; 2 &amp; 1 \ 2 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{-10 \alpha^6 + 8 \alpha^5 + 6 \alpha^4 + 16 \alpha^3 - 14 \alpha^2 - 24 \alpha + 18}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$d_3 = \begin{pmatrix} 2 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{-16 \alpha^6 + 48 \alpha^4 - 48 \alpha^2 + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
</tbody>
</table>

Note that the two distinct designs $d_1$ and $d'_1$ have the same $c_{d11}$. Utilizing MAPLE, we have compared $c_{d11}$ values of all $3 \times 3$ designs. It follows that $\max_{\alpha \in (-1,1)}(c_{d11} = c'_{d11}, c_{d211}, c_{d311}) \geq c_{d11}$, for all other $d \in D(2, 3, 3)$. Hence the values of $\alpha$ and the corresponding optimal design can be determined by comparing $c_{d11}$ of the above four designs. The values of $\alpha$ for which $c_{d11}$ or $c'_{d11}$ is greater than $c_{d211}$ and $c_{d311}$ are determined by solving the inequalities

$$\frac{-16 \alpha^4 + 32 \alpha^3 - 32 \alpha + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)} > \frac{-10 \alpha^6 + 8 \alpha^5 + 6 \alpha^4 + 16 \alpha^3 - 14 \alpha^2 - 24 \alpha + 18}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}$$

and

$$\frac{-16 \alpha^4 + 32 \alpha^3 - 32 \alpha + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)} > \frac{-16 \alpha^6 + 48 \alpha^4 - 48 \alpha^2 + 16}{(3 - \alpha)^2(1 - \alpha)^2(1 - \alpha^2)}.$$
The above two inequalities are satisfied for all $\alpha \in (-1, -1/5)$. Hence $d_1$ and $d'_1$ are optimal under (1.1) for all $\alpha \in (-1, -1/5)$. In a similar fashion, are obtained the values of $\alpha$ for which $d_2$ and $d_3$ are optimal. These values of $\alpha$ and the corresponding optimal (maximal $c_{d11}$) designs are reported in Table 2 below.

Table 2. Optimal $3 \times 3$ designs

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Optimal design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-1, \frac{-1}{5})$</td>
<td>$d_1 = \begin{pmatrix} 2 &amp; 2 &amp; 1 \ 2 &amp; 2 &amp; 1 \ 1 &amp; 1 &amp; 2 \end{pmatrix}$, \hspace{1em} $d'_1 = \begin{pmatrix} 1 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(\frac{-1}{5}, \frac{2\sqrt{7}-5}{3})$</td>
<td>$d_2 = \begin{pmatrix} 2 &amp; 2 &amp; 1 \ 2 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 2 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(\frac{2\sqrt{7}-5}{3}, 1)$</td>
<td>$d_3 = \begin{pmatrix} 2 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 1 \ 2 &amp; 1 &amp; 2 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

2.2. Optimal $3 \times 4$ designs

Optimal $3 \times 4$ designs are obtained by comparing $c_{d11}$ of all possible $3 \times 4$ designs. In this case, we have found two designs that maximize, depending on $\alpha$, the $c_{d11}$ element of the information matrix $C_d$. The values of $\alpha$ and the corresponding optimal $3 \times 4$ designs are in Table 2.

Table 2. Optimal $3 \times 4$ designs

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Optimal design</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-1, 0)$</td>
<td>$d_4 = \begin{pmatrix} 1 &amp; 1 &amp; 2 &amp; 2 \ 1 &amp; 1 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 1 &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$(0, 1)$</td>
<td>$d_5 = \begin{pmatrix} 1 &amp; 2 &amp; 1 &amp; 2 \ 2 &amp; 1 &amp; 2 &amp; 1 \ 1 &amp; 2 &amp; 1 &amp; 2 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
The $c_{d11}$ for designs $d_4$ and $d_5$ are as follows.

$$c_{d411} = \frac{-8\alpha^{10} + 64\alpha^8 - 176\alpha^6 + 224\alpha^4 - 136\alpha^2 + 32}{(3 - \alpha)(1 - \alpha)^2(1 - \alpha^2)^2(4 - 2\alpha)}$$

$$c_{d511} = \frac{-8\alpha^9 + 40\alpha^8 - 72\alpha^7 + 40\alpha^6 + 72\alpha^5 - 168\alpha^4 + 104\alpha^3 + 56\alpha^2 - 96\alpha + 32}{(3 - \alpha)(1 - \alpha)^2(1 - \alpha^2)^2(4 - 2\alpha)}.$$  

The values of $\alpha$ in Table 2 above are determined by solving the inequality $c_{d411} > c_{d511}$.

### 2.3 Optimal 3 × 5 designs.

The procedure used for the determination of optimal 3 × 5 designs is similar to that of 3 × 3 and 3 × 4 designs. Optimal 3 × 5 designs are obtained by comparing $c_{d11}$ of all 3 × 5 designs. The distinct designs that are found to be optimal are reported in Table 3.

#### Table 3. Optimal 3 × 5 designs

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Optimal Designs</th>
<th>$c_{d11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-1, -0.05561673968]$</td>
<td>{$\begin{pmatrix} 1 &amp; 1 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 2 &amp; 2 \ 2 &amp; 2 &amp; 1 &amp; 1 \ 1 &amp; 1 &amp; 1 &amp; 2 \ 2 &amp; 2 &amp; 2 &amp; 1 \ 1 &amp; 1 &amp; 1 &amp; 2 \ 2 &amp; 2 &amp; 2 &amp; 1 \ 2 &amp; 2 &amp; 2 &amp; 1 \end{pmatrix}$}</td>
<td>$\frac{48 - 160\alpha + 176\alpha^2 - 16\alpha^3 - 144\alpha^4 + 160\alpha^5 - 80\alpha^6 + 16\alpha^7}{(3 - \alpha)(5 - 3\alpha)(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$[-0.05561673969, 0)$</td>
<td>$\begin{pmatrix} 2 &amp; 2 &amp; 1 &amp; 1 \ 2 &amp; 1 &amp; 1 &amp; 2 \ 1 &amp; 2 &amp; 1 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{50 - 130\alpha + 74\alpha^2 + 78\alpha^3 - 130\alpha^4 + 74\alpha^5 - 2\alpha^6 - 22\alpha^7 + 8\alpha^8}{(3 - \alpha)(5 - 3\alpha)(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$(0, 0.05414365096]$</td>
<td>$\begin{pmatrix} 2 &amp; 1 &amp; 2 &amp; 1 \ 1 &amp; 2 &amp; 1 &amp; 1 \end{pmatrix}$</td>
<td>$\frac{50 - 40\alpha - 106\alpha^2 + 52\alpha^3 + 98\alpha^4 - 8\alpha^5 - 54\alpha^6 - 4\alpha^7 + 12\alpha^8}{(3 - \alpha)(5 - 3\alpha)(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
<tr>
<td>$[0.05414365097, 1)$</td>
<td>$\begin{pmatrix} 2 &amp; 1 &amp; 2 &amp; 1 \ 1 &amp; 2 &amp; 1 &amp; 2 \end{pmatrix}$</td>
<td>$\frac{48 - 160\alpha^2 + 192\alpha^4 - 96\alpha^6 + 16\alpha^8}{(3 - \alpha)(5 - 3\alpha)(1 - \alpha)^2(1 - \alpha^2)}$</td>
</tr>
</tbody>
</table>
The first four designs in Table 3 above have the same $c_{d11}$ and hence are equally good. The values of $\alpha$ in column one are determined by comparing the $c_{d11}$ values reported in third column under $c_{d11}$.

Note that the optimal $p \times q$ design (with $p \leq q$) for two treatments, when $\alpha = 0$ (errors are uncorrelated) and both $p$ and $q$ are odd, uses treatment one $p(q - 1)/2$ times and treatment two $p(q + 1)/2$ times, see Morgan and Uddin (1993). Thus the optimal designs with uncorrelated errors require that the two treatment replications differ by $p$. However, this is not the case for our optimal designs with large $|\alpha|$, see the designs in Tables 1 and 3 for large $|\alpha|$. Here the difference between the replications of two treatments is one, a criterion often preferred by practicing statisticians.

We have determined only $3 \times 3$, $3 \times 4$ and $3 \times 5$ optimal row-column designs for two treatments. It would be unwise to make any recommendation for all $p \times q$ designs based on these three designs. However, we suspect that the treatment allocation patterns found here, if extended to $p \times q$ designs, will give optimal $p \times q$ designs especially for large $|\alpha|$. For example, a design in which no treatment is neighbored by itself in rows and in columns is expected to be optimal for large positive $\alpha$.

Acknowledgement

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4. References


