DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

TERTIARY MATHEMATICS EDUCATION RESEARCH AND ITS FUTURE

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August 1999

No. 1999-6

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Tertiary Mathematics Education Research and Its Future

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1. What is Mathematics Education Research at the University Level?

Tertiary mathematics education research is disciplined inquiry into the learning and teaching of mathematics at the university level. It can be conducted from an individual cognitive perspective or from a social perspective of the classroom or broader community.[1] It can also coordinate the two, providing insight into how the psychological and social perspectives relate to and impact one another.

In the case of individual cognition, one wants to know how students come to understand aspects of mathematics, limit or linear dependence, or how they develop effective mathematical practices, good problem-solving skills, the ability to generate reasonable conjectures and to produce proofs. What goes on in students' minds as they grapple with mathematics and how might we influence that positively? More specifically, what difficulties do students have with the concept of limit? Does the everyday notion of speed limit as a bound present a cognitive obstacle? Does the early introduction of monotone increasing sequences constitute a didactic obstacle? Are there some, as yet neglected, everyday or school-level conceptions that university mathematics teachers might effectively build on? What is the influence of affect, ranging from beliefs through attitude to emotion, on effective mathematical practice? What roles do intrinsic and extrinsic motivational factors play?

From a social perspective, whether of a single classroom or some broader community, one seeks information on how social interactions affect the group as well as the individuals involved. For example, how might one change the classroom culture so students came to view mathematics, not as passively received knowledge, but as actively constructed knowledge? Or, how might one restructure an entire curriculum to achieve this effect? What are the effects of various cooperative learning strategies on student learning? What kinds of interactions are most productive? Are some students advantaged while others are disadvantaged by the introduction of cooperative learning? Which students succeed in mathematics? Which students continue in mathematics and why? **[Note to editors: Here there might be a cross-reference to the Forgasz & Leder paper in these volumes.]** What is the effect of gender, race, or social class upon success in mathematics? In coordinating the psychological and social perspectives, any of the above questions might be asked, along with inquiry into the reflexive relationship between the two perspectives. For example, how does an individual's contribution affect a whole class discussion and conversely?

Because mathematics education research investigates both individuals and groups in the process of learning mathematics, it has adapted methods from a variety of social sciences ranging from cognitive psychology to anthropology. Yet, because the inquiry is domain specific, i.e., often concerns questions directly involving the understanding of mathematical concepts, education researchers need to be well versed in that underlying mathematics.

Whether at the university level, or at the school level, mathematics education research is a relatively young field with an applied, or an applicable, character. Interesting questions, some suitably modifiable for investigation, arise when one encounters learning difficulties in the classroom. For example, are one's remedial college algebra students handicapped because they do not understand the notation? Are students of real analysis having difficulty because they are unable to generate suitable generic examples on which to build proofs? [Cf. Dahlberg & Housman, 1997.] Would visualization help analysis students construct proofs? [Cf. Gibson, 1998.]

Despite often arising from such everyday questions about the practice of teaching, mathematics education research does not often provide immediately applicable prescriptive teaching information, rather it provides general guidance and hints that might help with teaching and curriculum design. In a recent article, Hiebert (1999) pointed out that, while mathematics education research can inform us, it cannot tell us which curricula or pedagogy are "best" because such decisions necessarily involve value judgments about what students should know. For example, in this age of technology, when engineering students use computer algebra systems such as *Mathematica* or *Maple,* research cannot tell us which calculus and matrix algebra computations students should be able to perform flawlessly by hand. It can, however, inform us on the extent to which certain curricula, once implemented, have succeeded in attaining their goals.

A great variety of topics has been, and could be, investigated. These include aspects of mathematics (functions, analysis, proofs), mathematical cognition (problem solving, students' alternative conceptions), psychological factors (motivation, affect, visualization), teaching methods (lecturing, cooperative learning, uses of technology and writing), change (individual teacher and institutional), programs (new and existing), and culture (gender, equity, classroom culture, cross-cultural comparisons). The nature and size of studies varies from case studies of individuals to large-scale studies of hundreds or thousands of students (Schoenfeld, 1994). [For an overview of results at university level, see Dreyfus, 1990; Tall, 1991, 1992; Selden & Selden, 1993.]

Being a social science, mathematics education research cannot provide results having the character of certainty found in mathematics. Even very carefully conducted observations can only suggest general principles and yield evidence, rather than proofs. Thus, the corroboration of results by subsequent studies is important. Furthermore, the discovery of new results can sometimes make what previously appeared firmly established, less so later. Nevertheless, research in mathematics education shares a powerful "pyramiding" characteristic with other sciences; results rely on careful observations, are separated from the investigators' opinions, and are subject to community scrutiny, so that subsequent work can be based on them.

2. Making Progress in Tertiary Mathematics Education Research: "Pushing the Field Forward"

When a field, such as tertiary mathematics education research, is beginning, it is perhaps appropriate to survey the landscape to "see what's out there" by gathering many, sometimes rather isolated, bits of information. There is room for studies of misconceptions, as well as case studies of exemplary teachers or programs. Just as in mathematics, to get informative results, one needs to investigate interesting, yet accessible, questions. As Schoenfeld (1996) says in reference to education research, “The hard part of being a mathematician is not solving problems; it’s finding one that you can solve, and whose solution the mathematical community will deem sufficiently important to consider an advance. . . . In any *real* research [in particular, education research], the bottleneck issue is that of problem identification – being able to focus on problems that are difficult and meaningful but on which progress can be made.”[2]

A criterion often employed by reviewers and editors of mathematics education research journals for the selection of manuscripts is that they should "push the field forward," that is, establish something new and worth knowing. While some purely observational studies, for example, ethnographic reports, are seen as informative, increasingly it is important to place studies within the established body of research, citing how they fit in and extend or modify the work of others. [Cf. Hanna, 1998; Lester & Lambdin, 1998.]

Indeed, it is widely thought that to make significant progress, there is a need for theory building. Thus information is often gathered, analyzed, interpreted, and synthesized within an explanatory structure, usually called a *theoretical framework*, that provides some coherence, and perhaps even some predictive power, to the results.[3] While it is much too early for a "unified theory" of mathematics learning even at the elementary school level where more is known, there are concepts and theoretical frameworks which have proved useful in both making and explaining observations at the tertiary, and other, levels.

Research reports are expected to mention which of these concepts and theoretical frameworks are being used. That is, the researcher's assumptions, as well as the questions that were investigated, should be made clear. Beyond that, research results should be based on evidence (from data) and an analysis. [Cf. Lester & Lambdin, 1998.] Findings are also often required to be generalizable, that is, reports should provide enough detail about students, teaching, tests, and the like, so readers, who may be teachers of mathematics, can gauge how similar their situation is to the one being described, and hence, judge whether it has relevance for them.

2.1 Some Basic Concepts and Theoretical Frameworks Used in University Level (and Other) Mathematics Education Research

Philosophical views play a fundamental role in the perspectives researchers decide to take in their work and in the kind of research they do. Constructivism, the idea that individuals actively construct their own knowledge, can be traced back to Piaget and beyond and leads to emphasizing an individual cognitive perspective. In contrast, those taking a sociocultural view, emphasize the idea that culture mediates individual knowledge through tools and language, an idea that has roots in Vygotsky's work and leads to taking a social perspective.[4] However, such philosophical views apply to all knowledge, not just to mathematics and are often not particularly conspicuous in the research findings themselves.

Mathematics education research, being domain specific, has developed its own ways of conceptualizing mathematics learning. Indeed, Robert B. Davis, who was a long-time editor of the *Journal of Mathematical Behavior,* pointed out that one of the major contributions of mathematics education research has been to provide new conceptualizations and new metaphors for thinking about and observing mathematical behavior (Davis, 1990). It is very difficult to notice patterns of behavior or thought without having names (and the corresponding concepts) for them. As is often said of other empirical disciplines, one needs a lens (theoretical framework) with which to focus on (frame) what one is seeing.

2.1.1 Concept Definition versus Concept Image

The mismatch between concepts as stated in definitions and as interpreted by students is well-known to those who teach university level mathematics. The terms *concept definition* and *concept image* were introduced into the mathematics education literature to distinguish between a formal mathematical definition and a person's ideas about a particular mathematical concept, such as function. An individual's concept image is a mental structure consisting of all of the examples, nonexamples, facts, and relationships, etc., that he or she associates with a concept. It need not, but might, include the formal mathematical definition and appears to play a major role in cognition. [Cf. Tall & Vinner, 1981.] Furthermore, while contemplating a particular mathematical problem, it might be that only a portion of one's concept image, called the *evoked concept image*, is activated. These ideas make it easier to understand and notice various aspects of a student's thinking, for example, a student who conceives of functions mainly graphically or mainly algebraically without much recourse to the formal definition. A teacher or a researcher can investigate what sorts of activities might encourage students to employ the definition when that is the appropriate response, as in making a formal proof. It might also be helpful to investigate how students develop their concept images or how such images affect problem-solving performance.

2.1.2 Obstacles to Learning

One set of related ideas that has proved powerful is that of *epistemological, cognitive, and didactic obstacles*.[5] When applied to the learning of mathematics, these refer, respectively, to obstacles that arise from the nature of particular aspects of mathematical knowledge, from an individual's cognition about particular mathematical topics, or from particular features of the mathematics teaching. An obstacle is a piece of, not a lack of, knowledge, which produces appropriate responses within a frequently experienced, but limited context, and is not generalizable beyond it (Brousseau, 1997).

Regarding the first of these, sometimes a particular mathematical concept such as that of function, is fraught with inherent difficulties that can impede its full understanding until a radical reconceptualization has occurred. Using the historical development of function as a guide, it has been found that one epistemological obstacle that students need to overcome is the idea of function as expression, just as was the case with Euler (Sierpinska, 1992).

An example of a didactic obstacle -- one that can be traced back to an aspect of teaching -- is the geometric definition of tangent line, often taught to junior secondary students, as a line that touches a circle at precisely one point and is perpendicular to the radius at that point. In subsequent calculus/analysis courses, students must reconstruct, often with considerable difficulty, this idea more generally so that a tangent line to a curve at a point is the limit of secants and its slope is the value of the derivative at that point (Artigue, 1992). Other reconstructions -- of the real numbers, of integrals, of limit -- are necessary as students proceed from the intuitive and algebraic view of calculus presented at secondary school to the more formal presentations at university. **[Note to editors: Here it would be good to insert a cross-reference to the plenary of Artigue -- the part on reconstructions.]**.

2.1.3 Views of Concept Development

Another kind of distinction that has proved productive for tertiary mathematics education investigations has been the *action-process-object-schema* (APOS) view of an individual's mental construction of a concept like that of function.[6] In the APOS perspective, a function is first experienced via concrete actions, for example, through actually computing $x^2 + 1$, thought of as "first square a number, then add 1," for particular values of x. Only subsequently, may it become possible for an individual to conceive of a function as an input-output process, without actually undertaking such computations. At some still later stage, instances of the concept may come to be seen as objects in their own right, that is, as things which can be acted upon, say by a differential operator (Breidenbach, et al, 1992).

Somewhat similarly, Sfard (1991) has described an individual's journey from an *operational* (process) to a *structural* (object) conception as reification. She considers these to be dual modes of thinking and observes that previously constructed structural conceptions (objects) can be used to build new operational conceptions (processes). Furthermore, a single mathematical notation can reflect both a process and object conception; $2+3x$ stands both for the process of adding 2 to the product of 3 and x, as well as for the result of that process. Tall (1991, p. 254) coined the term, *procept*, to capture the process-concept (i.e., process-object) nature of many mathematical ideas and their notation. In order to be mathematically flexible, an individual seems to need both the process and object views of many concepts and the ability to move between these views when appropriate.

2.1.4 The Role of Mathematical Definitions as Contrasted with Everyday Definitions

A further idea that appears to be very useful, but which may not yet be widely found in the literature, is the distinction between *synthetic* and *analytic definitions*.[7] Synthetic definitions are the everyday definitions that are commonly found in dictionaries -- they are descriptions of something that already exists. They are often incomplete, yet redundant. For example, on the crudest level, one can define a democracy as a form of government in which the people vote. However, additional properties might better characterize governments normally regarded as democracies. It is often unclear when such everyday definitions are "complete" or whether attention to all aspects of them is essential for their proper use. Analytic definitions, by contrast, bring concepts into existence -- the concept is whatever the definition says it is, nothing more and nothing less. Thus, for example in a graduate course, one usually defines a semigroup as a set together with a binary associative operation on it and immediately begins to deduce properties about semigroups. One cannot safely ignore any aspects of such definitions. Many of the difficulties that university students have with formal mathematics might well be viewed as stemming partly from an unawareness that mathematical definitions tend to be analytic, rather than synthetic, or from an inability to handle formal mathematics even when a difference in the two kinds of definitions is perceived.

2.1.5 How Might These Concepts Might be Used by University Teachers of Mathematics?

Such ideas (e.g., concept image, epistemological obstacle, action-process-object-schema, synthetic vs. analytic definitions) help frame, not only research, but also discussions of teaching and learning toward more insightful, and ultimately, more productive ends. They help one view students' attempts at mathematical sense-making and understanding as somehow hindered by their current, somewhat limited, conceptualizations -- instead of merely emphasizing that university students don't do their homework, aren't motivated, or are just plain lazy (some of which may also be true).

While it is perhaps too soon to expect such ideas to have moved far beyond the mathematics education research community and it is hard to gauge the practical effects of anyone's use of new concepts, there are a few hints that some teachers and authors are finding them useful. For example, not long ago an author of undergraduate mathematics textbooks indicated that he finds the idea of concept image useful in his teaching and writing.

In general, the pedagogical challenge is to figure out how to help students come to genuine mathematical understanding. Which instructional efforts might be more productive of genuine mathematical understanding? Here various techniques have been tried -- computer activities that provoke students to reflect on mathematical situations (and to explicitly construct actions, processes, objects, and schemas), group projects that require them to grapple with mathematical ideas, and process writing to help students clarify their mathematical thoughts (by explicitly describing the evolution of their thinking whilst wrestling with problem situations). For example, when there was concern about the pass rate of U.S. university calculus students, the National Academy of Science convened a National Symposium in 1987 which resulted in a concerted effort on the part of the National Science Foundation to promote calculus reform.[8] Some combination of the above pedagogical strategies was included in many of the resulting calculus reform projects (Tucker, 1990), but very few of these were based on research ideas such as those mentioned above.

2.2 Some Theories of Instructional Design

While the above ideas can prove helpful to university teachers of mathematics desiring to understand "where their students are coming from," it would also be useful for mathematics education research to inform the development of curricula. Although there are relatively few mathematics education researchers working at the university level worldwide and information is really just beginning to accumulate, there have been some efforts at curriculum design using the results obtained so far. Here are four examples, the last of which did not arise from the research literature, but is in considerable agreement with it. All four teach through student-solved problems and avoid providing worked, template examples.

2.2.1 APOS Theory and the ACE Teaching Cycle

The learning of many university level mathematical topics has been investigated using the APOS (Action-Process-Object-Schema) theory and instructional sequences have been designed reflecting it. Envisioned as an iterative process, this instructional design process begins with a theoretical analysis, called a *genetic decomposition*, of what it means to understand a concept and how that understanding might be constructed or arrived at by the learner. This initial analysis is based on the researchers' understanding of the concept and on their experiences as learners and teachers of mathematics. This leads to the design of instruction, which is subsequently implemented and observed. Data is gathered and analyzed, and this analysis leads to revisions of both the theoretical analysis and the instructional design.

Since this approach views the growth of mathematical understanding as highly non-linear -- with students developing partial understandings, often repeatedly returning to the same concept -- the instructional approach consists of "an *holistic spray*, a variation of the standard spiral method." Students are intentionally put into disequilibrating situations (in which they see their lack of understanding) and, individually or in cooperative groups, they try to make sense of these situations, e.g., by solving problems, answering questions, or understanding ideas. A particular strategy used is the *ACE Teaching Cycle*, consisting of three components: Activities, Class discussion, and Exercises. The activities often involve extensive teamwork on ISETL[9] computer programming tasks, whose design is based on the proposed genetic decomposition of a particular concept. The intent is to provide students with experiences that promote the development of that concept and upon which they can build in the forthcoming discussions. In the instructor-led class discussions which usually take place on a subsequent day, the students again work in teams, but this time on paper-and-pencil tasks based on the computer activities. This is followed by relatively traditional out-of-class exercises to be worked individually or in teams; their purpose is to reinforce the mathematics learned. Because this pedagogical strategy is somewhat unconventional, its designers have found it necessary to create textbooks to support it; this has been done for discrete mathematics, precalculus, calculus, and abstract algebra. These textbooks do not contain template problems and no answers are given to the exercises; the students are encouraged to investigate mathematical ideas for themselves and are allowed to read ahead in their own or other textbooks.[10] [Cf. Asiala, et al, 1996. **Note to editors: Also put a cross-reference to the Dubinsky and McDonald paper in these volumes.**]

2.2.2 Didactical Engineering and the Method of Scientific Debate

Another approach to research and curriculum design is *didactical engineering*, a method favored by many French mathematics education researchers. Based on both Chevallard's *theory of didactical transposition* and Brousseau's *theory of didactical situations*[11], didactical engineering is a form of design activity which employs the results of research and which is devoted to the elaboration of pedagogical strategies and curriculum materials specifically for mathematics. In this method, the teacher's job is to present students with judicious "problems" intended to provoke the students into accepting them and seeking solutions on their own. That is, the teacher proposes mathematical situations to actively engage the students. Learning is the result of students' adaptation to these mathematical situations (Brousseau, 1997).

Artigue has used didactical engineering in an instructional approach to differential equations (for first-year university students of mathematics and physics) that, from the beginning, coordinated algebraic, numerical, and graphical approaches with the solution of an associated differential equation.[12] Using specially developed computer software, students first draw curves through the direction field of tangents, study isocline lines and possible phase portraits, and then justify the most likely graphs using algebraic or graphical arguments (Artigue, 1991).

The *method of scientific debate* is a somewhat similar French approach to conducting mathematics courses with the aim of having students act mathematically, rather than merely covering the syllabus. Students are encouraged to become part of a classroom mathematical community in which they propose conjectures and debate their relevance and truth. Students develop rational arguments to convince themselves and others. For the method be successful, the professor should refrain from revealing her or his opinion, should allow time for students to develop arguments, and should encourage maximum student participation. A renegotiation of the "didactic contract" is necessary so that students come to understand and accept their responsibilities as active participants in the knowledge-building process. [Cf. Legrand, 1993. **Note to editors: Also put a cross-reference to Legrand's paper in these volumes.**]

2.2.3 Realistic Mathematics Education and Local Instructional Theories

A third approach to the interaction between research and curriculum design, more often practiced at the elementary and secondary levels but recently being tried at the university level, is that of *realistic mathematics education*. From this perspective, students learn mathematics by mathematizing the subject matter through examining "realistic" situations, i.e., experientially real contexts for students that draw on their current mathematical understandings. In this approach, the problems precede the abstract mathematics, which emerges from the students' collaborative work towards solutions. This approach goes back to Freudenthal and is favored by the Dutch school of mathematics education researchers. Curricula, as well as the instructional theory and its justification, are mutually developed and refined in a gradual, iterative process.

In this approach, curriculum design tends to be integrated with research, perhaps because it is difficult to predict how students will tackle problems for which they have no model solutions. Beginning with realistic mathematics education as the global perspective, the aim is to develop *local instructional theories*, whether these be for the teaching and learning of fractions or differential equations. In a manner somewhat analogous to the cycle of development mentioned in 2.2.1, this developmental process begins by positing hypothetical learning trajectories, along with a set of instructional activities. After an instructional sequence has been implemented and observed, researchers engage in retrospective analysis that leads to refinement and revision of the conjectured learning trajectory. Three heuristics are used in designing curricula: (1) the *reinvention principle*, whereby students are guided to construct at least some of the mathematics for themselves, (2) *didactic phenomenology*, whereby researchers analyze practical problems as possible starting points for the reinvention process, and (3) the construction of *mediating, or emergent models* of students' informal knowledge and strategies in order to assist students in generalizing and formalizing their informal mathematics. [Cf. Gravemeijer, 1998; **Note to the editors: Here there could also be a cross-reference to the conference paper of Chris Rasmussen and Karen King.]**

2.2.4 The Moore Method of Teaching

This distinctive method of teaching has developed into an informal method of curriculum design and has evolved naturally without calling on research or theory in mathematics education. However, although it arose prior to, and independently of, didactical engineering and the work of Brousseau, some of its aspects are derivable from that work. There is a renegotiation of the didactic contract, a teaching through carefully selected problems (usually requiring the construction of proofs), "devolution" of the problems to the students (i.e., transferring to them an interest in, and an obligation for, the production of proofs), and "adidactic situations" requiring the students to solve the problems on their own. Students are presented many opportunities (situations) for facilitating personal knowledge construction, but also construct much of the actual mathematics, usually in the form of proofs, themselves.

The method developed out of the teaching experiences of a single accomplished U.S. mathematician, R. L. Moore, and has been continued by his students (several of whom went on to become presidents of the American Mathematical Society or the Mathematical Association of America) and their mathematical descendents. It has been remarkably successful in producing research mathematicians, but has also been used in undergraduate university classes. In many versions, students are given definitions and statements of theorems or conjectures and asked to prove them or provide counterexamples. The teacher provides the structuring of the material and critiques the students' efforts. Since apparently no one knows how to effectively tell someone else how to prove a theorem or even how to very usefully explain to a novice what constitutes a proof, students just begin. This sounds a little like throwing someone into a lake in order that he or she learn to swim. However, once a student proves the first small theorem (stays afloat mathematically), it is often possible for her or him to make very rapid progress.

Moore method courses could provide interesting opportunities for research in mathematics education and the method would probably benefit from an analysis in terms of didactical engineering. Even the teaching of Moore himself has not been well researched.[13] However, there is currently an effort to document both the life and teaching method of Moore and to encourage "discovery learning."[14] The project is based at the University of Texas, where Moore taught for many years. It is called the R. L. Moore Legacy Project [http://www.discovery.utexas.edu/rlm/].

2.2.5 What Can Theories of Instructional Design Do for a Teacher of Tertiary Mathematics?

The above theories are concerned not only with producing curriculum materials, such as textbooks or software, for an individual course[15], but also with classroom practices, i.e., with how a course is taught and what goes on in a classroom between students and students and between teacher and students. Such theories may provide ways of approaching long-standing teaching problems, four examples of which are discussed below.

(1). A teaching problem that seems fairly widespread and recurrent concerns what to do about complaints from client disciplines, such as engineering, chemistry, or business, that their students cannot apply what they were supposed to have learned in mathematics courses. A natural response is to incorporate more applications into mathematics courses, but there are often too many different applications. As a result, students often need to handle some applications they have not seen before, but even "successful" students can have remarkable difficulty solving even moderately novel problems (Selden, Selden & Mason, 1994). One requirement for solving novel problems is a solid understanding of the mathematical concepts involved. The APOS theory and ACE teaching cycle seem especially useful for concept development and have worked well with concepts like function and quotient space. However, those who have struggled to prove theorems, all of whose concepts were familiar, will most likely agree that conceptual grasp is not the whole story in solving moderately novel problems.

(2). Motivating students is another perennial problem. It is often thought that starting with real world problems helps, but sometimes this can lead students to see mathematics as overly utilitarian and of little intrinsic interest. Didactical engineering, the method of scientific debate, and realistic mathematics education have particular promise for motivating students. They consider things like the didactic contract and the culture of the classroom in order that students will come to see revised classroom practices, such as the necessity to give reasons, as "normal" and also build more interest in mathematics itself.

(3). Most experienced undergraduate mathematics teachers can easily identify a number of topics with which many students will have difficulties. These include the concept of variable, working with "split domain" functions, limits, the Fundamental Theorem of Calculus, sequences and series, the ideas of proof and vector space, and the open cover definition of compactness. Methods arising from APOS theory are particularly concerned with helping students construct mathematical concepts and are likely to be helpful in teaching such topics.

(4). Finally, it is probably an annoyance to many teachers of tertiary mathematics that their students, and indeed most of the general public, have very little idea what mathematics is about. This is not just a matter of inconvenience to mathematicians -- rather a population that understands little of mathematics or science is likely to make poor public policy decisions. Theories that are concerned with things like the didactic contract or the culture of the classroom are particularly suited to encouraging students to move towards a more accurate view of mathematics, because explanation, justification, and proving come to be seen as a normal part of mathematics classes.

More research and more efforts at research-based curriculum design would certainly be beneficial. With this in mind, we will now concentrate on who does tertiary mathematics education research, how they might be trained, where they might find their "academic home," and other issues related to the production of research results in, and ultimately curriculum materials for, tertiary mathematics education.

3. Who Does Tertiary Mathematics Education Research?

While there is a substantial amount of research in mathematics education at the school level (Grouws, 1992), the amount at the tertiary level is still modest. Some tertiary studies, e.g., those investigating the effects of gender or the kinds of students who succeed in mathematics, have been conducted by mathematics education researchers without a particularly strong background in tertiary mathematics. However, many studies will require the researchers, or at least some members of a research team, to have an extensive knowledge of tertiary mathematics itself.

3.1 Getting into the Field

Although a few universities have developed graduate programs with a specialty in tertiary mathematics education leading to the Ph.D., many current researchers have come from the ranks of mathematicians. For example in France, someone interested in learning to conduct such research typically already has a teaching position and joins a research team for 2-3 years. Others have made the transition themselves. In the U.S., there has been an effort by the Research in Undergraduate Mathematics Education Community (RUMEC) to mentor interested mathematicians into the field; this group is interested in both doing and promoting research in tertiary mathematics education. [Cf. http://www.maa.org/data/features/rumec.html]. It is expected that this mentoring work will be continued by the recently formed Association for Research in Undergraduate Mathematics Education (ARUME), which is affiliated with the Mathematical Association of America (MAA).

Where will such researchers come from in the future? Retrained mathematicians? Post-graduate departments of mathematics education? What should such researchers know? It seems apparent that they should know a substantial amount of both mathematics and mathematics education, but what topics or courses are most appropriate to their preparation? Would the development of researchers and the furtherance of research be aided by the publication of a "research agenda" series (e.g., Charles & Silver, 1989), much like the National Council of Teachers (NCTM) did for mathematics education research at the school level?

3.2 Preparation of Future Tertiary Mathematics Education Researchers

Current employment opportunities, combined with the need for more tertiary mathematics education research, suggest some desirable features of Ph.D. programs. In the U.S. and some other countries, new mathematics education Ph.D.'s can find employment in mathematics departments, provided they have sufficient knowledge of mathematics -- usually the same knowledge as a mathematics Ph.D. except for the dissertation specialty. Often they are expected to teach mathematics to preservice teachers, so a knowledge of school level, as well as tertiary level, mathematics education research is important.

Because mathematics education research varies greatly in both the methodology used and the topics investigated, learning about these from a sequence of specially taught classes, rather than independent reading, would probably be more efficient. As for the dissertation, its traditional form in education seems somewhat problematic in that it sometimes does not adequately prepare new Ph.D's to publish research results, even those contained in the dissertation itself (Duke & Beck, 1999). Fortunately, this is not always so. The research results from some mathematics education dissertations are rewritten and condensed for journals, and occasionally dissertations are even published as monographs. Since there is a need for more tertiary mathematics education research, Ph.D.'s should know how to produce and publish it. This suggests emulating Ph.D. dissertations that have led to research publications or a dissertation model closer to that of mathematics; that is, a dissertation that consists of one or more publishable papers integrated with enough background material to be "self-contained."

Beyond graduate programs in tertiary mathematics education, what less formal training might be useful for would-be researchers? Intensive graduate level summer seminars might help; in France, there is an extensive Research Summer School (Didactique des Mathématiques), which might be mimicked in other countries. Perhaps editors of mathematics education research journals might target promising new researchers to review (referee) several manuscripts, thereby introducing them to the criteria for acceptance. Normally, editors try not to overburden individuals, but reviewing papers can be very educational, especially where reviewers are ultimately provided with the editor's and other reviewers' reports. The latter is standard practice for the *Journal for Research in Mathematics Education*.

Also, more mentoring programs, like those of the RUMEC group and ARUME might be beneficial. One or two-day short courses, such as those given at meetings of the American Mathematical Society (AMS) and the Mathematical Association of America (MAA), might be productive. In order to join a research team, as in the French model, individuals usually need financial support. Often, one's home university, will not finance extended full-time leaves. One promising program in the U.S. is the National Science Foundation's PSFMETE program, which supports Ph.D.'s in science, mathematics, engineering, and technology for 2-3 years to pursue projects in curriculum design supervised by an appropriate mentor at a major university.

3.3. The Placement of Tertiary Mathematics Education Researchers

Where will tertiary mathematics education researchers find their academic “home”? Although there are a number of Ph.D.-granting institutions around the world producing researchers in tertiary mathematics education, unless these individuals already have teaching posts in universities, it is not always clear who will hire them. In United Kingdom, with its current emphasis on research within the academic disciplines, it is almost impossible for new Ph.D.'s specializing in tertiary mathematics education research to obtain employment in university mathematics departments (Adrian Simpson, personal communication). In France, didactics is now considered a legitimate applied mathematics research specialty and tertiary mathematics education researchers are "nationally evaluated with the same criteria as other applied mathematicians;" yet this acceptance is still somewhat fragile (Artigue, 1998).

In the U.S., because of the large number of four-year liberal arts colleges and comprehensive state universities, most of whose mathematics departments teach preservice teachers, the problem of academic "home" seems less acute. However, only a few mathematics departments in major research universities currently hire persons whose primary research area is tertiary mathematics education although this may be changing. In the past, some mathematics departments have had members who became de facto specialists in teaching preservice teachers or lower-division and remedial students. An understanding seems to be developing that such teaching calls for specialized training best provided by knowledge of mathematics education and that those with such a background can also be expected to be publish. This trend should gradually open up new positions in the U.S. for researchers in tertiary mathematics education.

The problem of academic "home" is sometimes exacerbated by the low status accorded education research by some mathematicians, who may suspect that such research is either useless nonsense or an elaboration of common sense classroom practice. Those who specialize in mathematics education (didacticians) are sometimes seen as a "kind of submathematician who finds, in didactic research, a diversion from his or her lack of mathematical productivity" (Artigue, 1998). Of course, someone who has published a research paper in say, the *Journal for Research in Mathematics Education*, might view the matter differently. In Malaysia and China, mathematicians also seem to value only mathematics research, while denigrating mathematics education.

4.   Making the Results of Mathematics Education Research Available to Those Teaching Mathematics in Universities: The Application of Research to Teaching Practice

Mathematics education research cannot normally be expected to tell anyone precisely how to teach or to predict whether a teaching method or curriculum will be effective with particular students. What it can do is provide information useful in teaching and for curriculum development, that is, for what mathematicians have always done informally -- develop courses, textbooks, and teaching methods. It can give insights into students' intuitive views on limits, functions, logic, etc., which might either be obstacles to new knowledge or useful as bridges to new knowledge. It can help develop new pedagogical content knowledge, i.e., ways of teaching specific mathematical topics that arise from an understanding of both mathematics and pedagogy. It can help develop new pedagogical content knowledge, i.e., ways of teaching specific mathematical topics that arise from an understanding of both mathematics and pedagogy. It can bring out non-obvious barriers to changing teachers' beliefs or pedagogical techniques and a great deal else useful in teaching and curriculum design.

4.1 Dissemination: Getting the Word Out to Those Who Teach University Mathematics

In order to be useful, research results on the teaching/learning of university level mathematics needs to come to the attention of those who teach it, primarily mathematicians, but also including the ever-growing cadre of community college mathematics teachers, lecturers, tutors, etc. Currently, there are several ways of bringing research results to the mathematics community.

In the U.S. a start was made in that direction with the newsletter, *UME Trends*[16], which contained a column called Research Sampler that briefly described mathematics education research results applicable to the tertiary level. Although *UME Trends* has now ceased publication, the Research Sampler column continues in a modified form as part of the Teaching and Learning Section of *MAA Online* (www.maa.org). In addition, a few of the regular journals of the Mathematical Association of America (MAA), e.g., *The College Mathematics Journal*, have agreed to publish expositions of mathematics education research results from time-to-time. The effects of this, and of Web sites such as that of *MAA Online*, are likely to remain modest, because of the shortage of articles. Potential authors of such expository papers are usually the researchers themselves and many universities undervalue exposition as compared to research when it comes to tenure and promotion. It might be possible to alleviate the effect of this dearth of high quality exposition by establishing more links between existing websites.

Other avenues for dissemination include the annual meetings of ARUME with MAA, at which mathematics education research papers will be featured, along with an expository talk. However, at best this effort can only reach the few thousand mathematicians that attend such meetings. A similar role might be played in the U.K., by the Advanced Mathematical Thinking Working Group. Another modest start towards dissemination and recognition of the field, is the fact that *Zentralblatt für Didaktik der Mathematik (ZDM)* and *Mathematical Reviews (MR)* now both include coordinated categories for abstracting research articles in undergraduate mathematics education.

It would be especially beneficial to find ways to bring tertiary mathematics education research results to the attention of graduate students in mathematics, many of whom will take up teaching posts in universities. In the U.S., the Exxon-funded Project NEXT (New Experiences in Teaching) has given several hundred new mathematics faculty members the opportunity to meet and network at annual meetings of MAA, while also attending special workshops on technology and teaching that include some mention of tertiary mathematics education research results.

4.2 Integrating Research Results into Teaching Practice

The most effective way of bringing tertiary mathematics education research into teaching practice, seems to be via new research-based curricula. In Section 2.2 there are three examples of ways that research has been systematically integrated with curriculum design. In addition, the results of research can also be used in less systematic ways to inform teaching and curriculum development. For example, mathematics education researchers at San Diego State University in the U.S. have used research results in preparing a CD-ROM to assist university instructors of mathematics courses for pre-service elementary and middle school mathematics teachers. Several other such video/CD-ROM projects exist, but it is not clear whether these are targeted at those teaching in mathematics departments or in education departments. In general, however, the mathematics community has yet to make much use of research results in either teaching or curriculum design. Thus the need for enhanced dissemination in order to bring about long-range benefits.

4.3 Some Suggestions for Reaching University Mathematics Teachers

Perhaps team teaching, departmental seminars on teaching, or other local efforts could facilitate the incorporation of research results and generally improve pedagogy. One might try intensive summer workshops, such as that on Cooperative Learning in Undergraduate Mathematics Education (CLUME), or the Park City Mathematics Institute which brings together a combination of about two hundred assorted mathematicians, undergraduate and high school mathematics teachers, and mathematics education researchers for three intensive weeks, with opportunities for interaction between all groups. [Cf. http://vms.www.uwplatt.edu/~clume/;

http://www2.admin.ias.edu/ma/park.htm.] Such workshops and institutes tend to be expensive; the two mentioned here were funded by the U.S. National Science Foundation (NSF).

Currently, there are "research into practice" sessions (Wilson, 1993) at NCTM meetings; perhaps similar sessions at meetings of university mathematics teachers, such as those of the AMS/MAA, would be beneficial. These might include video clips from research, showing students engaged in mathematical problem solving, or teaching episodes; such clips often provide convincing visual evidence and "talking points" for interested, even skeptical, teachers.

It is important to have mathematics Ph.D. students, who will become tomorrow's university mathematics teachers, take research in tertiary mathematics education seriously. Perhaps, in those Ph.D. programs where coursework is required, one could insist that students take a course on tertiary mathematics education research, or even conduct a mini-research project on some aspect of students' mathematical thinking. One practical problem is: Who would teach such a course? Also, perhaps it would be helpful to develop a list of expository and other readings in tertiary mathematics education research, post it on the Web, and update it regularly.[17] If so, whose responsibility would that be?

5. Possible New Directions for Mathematics Education Research at the University Level

Clearly, the existing tertiary mathematics education research barely "scratches the surface." While some topics of interest in both secondary and tertiary teaching, like the function concept, have benefited from being considered by a number of researchers[18], other topics such as real analysis or the learning of post-graduate students in mathematics are just now beginning to be studied.

One fairly natural way to collect and refine research questions is to examine one's own teaching; for that to be successful, mathematics education researchers need to teach, not only preservice teachers as is often the case in the U.S., but also specialist and nonspecialist (e.g., engineering and business) students at the tertiary level.

Many areas such as students' learning, teaching and teacher change, problem solving and proofs, social structures like departments, views of mathematics, theoretical frameworks, and pedagogical content knowledge are ripe for further investigation.

5.1 Some Ideas that Might be Worth Pursuing

In his plenary address, Hyman Bass pointed to four areas of mathematics education, with some associated questions, that are critically in need of systematic research: the secondary/tertiary transition, instructional use of technology, university-level teaching, and the context of the university with respect to teaching. **[Note to editors: Here it would be good to cross-reference the plenary of Bass]**. Here is a potpourri of additional questions.

5.1.1 Beginning University Students: The Secondary/Tertiary Interface

In the U.S., many students entering junior colleges and comprehensive state universities are unprepared to take calculus, and much teaching occurs at the precalculus level. Some of these students are older, non-traditional students whose secondary mathematical preparation needs renewal. Would it be useful to catalogue the many, and possibly interacting, difficulties of these precalculus students? Lately, while teaching such courses, we noticed students who fell asleep during an examination, spent significant time murmuring "I hate math," and had difficulty reading and interpreting test questions, e.g., completely ignoring adjectives like "positive." In addition, some students are very weak in simple algebraic skills and cannot make "multi-level" substitutions. Although a great deal of work has been done on the learning of algebra at secondary school (Wagner & Kieran, 1989), many of these students have already passed secondary algebra courses, suggesting that remediating their difficulties might need special instructional treatments.

Indeed, even for successful secondary school graduates, there are a number of problems concerning the secondary/tertiary interface. Often the mathematics curriculum at secondary school encourages the development of informal, intuitive ideas, whereas university mathematics courses tend to be more formal and rigorous. Typical of the difficulties encountered is the necessity to reconstruct one's understanding of the real numbers to allow for the equality of 0.999... and 1 and to reconstruct one's notion of equality to allow two numbers a and b to be equal if $|a - b| < 1/n$, for all natural numbers n. **[Note to the editors: Here the might be a cross-reference to the plenary of Artigue.]** While some of the difficulties are well known, what to do about them is not.

5.1.2 Learning to Understand and Validate Proofs

Once students get beyond calculus, they move into courses that are more formal and often require them to construct and validate proofs. Validating a proof, i.e., reading it to determine its correctness, involves mentally asking and answering questions, inventing supplementary proofs, etc. (Selden & Selden, 1995). This validation process is part of the implicit curriculum and appears to be a principal way mathematicians learn new mathematics. But in normal circumstances, it is largely unobservable. Thus, both mimicking the validation process and conceptual reflections on it may be difficult for students. How does one learn this validation process? How are various kinds of prerequisite knowledge, e.g. logic, related to learning to construct proofs? Much of this prerequisite knowledge seems best learned in the context of making proofs and its contextualization is lost when it is linearized and taught prior to proofs -- despite considerable attempts to do so in the U.S. via "bridge courses."

From this viewpoint, the difficulties students have with proofs might partly be a "didactic obstacle," arising from separating prerequisite knowledge from the proof context. This idea is consistent with the success of the Moore method of teaching (described in 2.2.4), in which students invent proofs of a carefully structured sequence of statements, but are not explicitly or separately taught the prerequisite knowledge before starting to construct proofs. With the aid of hypertext, one might integrate, in a nonlinear way, learning to construct proofs with "just in time" knowledge of logic, functions, sets, etc.[19]

5.1.3 Teaching Service Courses for Non-specialists

As Artigue points out, much research at the tertiary level has, often implicitly, taken the view that universities train future mathematicians, whereas a large amount of university mathematics teaching occurs in "service courses" for "client disciplines," a trend that may well increase. **[Note to the editors: Here there might be a cross-reference to Lynn Steen's plenary.]**

There have been a few studies of how practicing professionals -- architects, biologists, bankers, nurses -- use mathematics, with the ultimate aim of improving the teaching of such courses. The classic view of mathematical modeling, which involves identifying and simplifying a problem, solving a decontextualized mathematical version, and mapping the solution back, does not agree with workplace experience. [Cf. Pozzi, Noss & Hoyles (1998); Smith, Haarer & Confrey, (1997); Smith & Douglas, (1997); Noss & Hoyles (1996)] More workplace studies of mathematics use, especially as they relate to curriculum development, would be helpful.

In addition, the teaching of mathematics to preservice elementary and secondary teachers comprises another large share of the courses taught in some mathematics departments. Because of its effect on the teaching of school level mathematics this aspect of tertiary mathematics education has been somewhat better studied and is often the subject of papers at conferences, such as those of PME and PME-NA.[20]

5.1.4 Aspects of Teaching Practice and Institutions that Affect Learning

What views of learning do tertiary mathematics teachers have and how do these affect their practice? Does a teacher's pedagogical knowledge closely resemble the kind of automated procedural knowledge that might be called upon in actual teaching practice? That is, can one predict, and ultimately change, moment-to-moment pedagogical strategies? (Schoenfeld; 1998.) In athletics, knowing how a game should be played is rather different from being able to play it.

The relationship of teaching practice and the mathematical and pedagogical beliefs of mathematics department members to the leadership and power structure of a department has not been well examined. Yet individual teacher change may depend significantly on such social structures. For example, we know a mathematician, tenured in a small department, who will try new teaching techniques on upper-division courses, but not on calculus because colleagues might disapprove.

Could some research be directed towards generating pedagogical content knowledge, e.g., how to teach the Chain Rule or an explanation of why some university students persist in adding fractions incorrectly? Such knowledge can be a major part of the preservice teacher curriculum, but there is a dearth of it at the tertiary level. Perhaps some mathematicians would be interested in discovering and analyzing pedagogical content knowledge by conducting small teaching experiments, thereby making a contribution without having to delve deeply into the theoretical aspects of mathematics education research.

5.1.5 Philosophical and Theoretical Questions

Views of mathematics arising from the current philosophical climate tend to treat mathematics as asocial or mental construct, and sometimes equate objectivity with social agreement (Ernest, 1998). This appears inconsistent with the ideas of many mathematicians who often see themselves as approaching some kind of abstract knowledge which is independent of time and place. Is there a synthesis of these two apparently contradictory positions which is compatible with both?

What are some promising directions to develop or extend theoretical frameworks? The action, process, object, schema (APOS) theory (Asiala, et al, 1996; Sfard, 1991) may not yet have reached its full potential. Concepts can be not only objects (e.g., topological spaces), but also properties (e.g., compactness) and activities (e.g., factoring). Are the last two of these learned in a way similar to the first? Or, to take another example, the classification of memory as long-term, short-term, and working seems a somewhat neglected, but promising framework. Might errors that students sometimes refer to as "dumb mistakes" be explainable in terms of working memory overload? Could the need to minimize working memory load and the possible resulting mistakes in validation (of proofs) help explain the terse, minimalist rhetoric of proofs?

Could Schoenfeld's (1985) large-grained analysis of problem solving, in terms of resources, control, heuristics, and beliefs, be expanded to include a finer-grained, mechanistic view of how ideas are brought to mind during problem solving? Could Pirie and Kieren's (1994) analysis of the growth of mathematical understanding, as applied to problem solving, be integrated with Schoenfeld's more large-grained perspective? In attempting to solve moderately novel problems, some students seem to have knowledge they cannot access, i.e., bring to mind (Selden, Selden, & Mason, 1994). However, good solvers of moderately novel problems seem to associate general classes of problems, e.g., optimization problems, with several remembered tentative initial strategies. Could such aspects of knowledge function to link recognition of a class of problems to the recall of knowledge that might be useful in their solution?

As technology permeates the curriculum of engineering and other students, questions of which mathematics to teach and how to teach it come to the fore. Students can often carry out tasks using software imbued with mathematics they haven't yet learned; thus learning to critique the solutions generated (e.g., recognizing the effect of varying parameters) is important. [Cf. **Note to editors: Here it would be good to cross-reference**

the paper of Kent & Noss in these volumes.] Research questions include: How does the design of technology-based service courses in mathematics shape students understandings of mathematics? What kinds of messages about mathematics do students receive in such courses?

5.2 Professional Research Organizations and Avenues for Publication of Research Results

In order for a thriving community of tertiary mathematics education researchers to develop and prosper, there need to be adequate opportunities for the presentation and publication of research results. Currently, the following journals and refereed publications will accept tertiary mathematics education research articles, but many publish research mainly at the school level:

- *Journal for Research in Mathematics Education*
- *Educational Studies in Mathematics*
- *Journal of Mathematical Behavior*
- *Proceedings of PME* and *PME-NA*
- *Recherches en Didactiques des Mathématiques*
- *Focus on Learning Problems in Mathematics*
- *For the Learning of Mathematics*
- *International Journal of Computers for Mathematical Learning*
- *Mathematical Thinking and Learning*.

However, the *Research in Collegiate Mathematics Education* volumes specialize in tertiary mathematics education research papers and function much like a journal. They appear in the Conference Board of the Mathematical Sciences series, Issues in Mathematics Education, published by the American Mathematical Society.

In addition, tertiary mathematics education researchers need a professional organization devoted to their interests. Some promising developments along this line are occurring. In January 1999, the Association for Research in Undergraduate Mathematics Education (ARUME) was formed in affiliation with the MAA; it is devoted to research in undergraduate mathematics education and its applications. Meeting in conjunction with the MAA provides an opportunity to interact with, and possibly influence, mathematicians. [Cf. http://www.maa.org/t_and_l/arume_ann.html.] In February 1998, the Advanced Mathematical Thinking (AMT) Working Group was established at the British Society for Research in Learning Mathematics (BSRLM). Its main aims include continuing to develop a psychology of advanced mathematical thinking, understanding how mathematicians and students think about advanced mathematics, and to provide better ways of teaching students. [Cf. http://www.soton.ac.uk/~amt.]

6. Conclusion

While answers, or even partial answers, to questions in tertiary mathematics education will not be easy to come by, there is plenty of room for a variety of research.[21] There is a need for researchers and for dissemination of research results, as well as for cooperation between mathematicians and mathematics education researchers at all levels.

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[1] There does not seem to be any one universally accepted approach to research in mathematics education. However, since the days of Frances Bacon, science has been seen as consisting of making observations, reporting them to others, formulating theories, and finally testing them. Some mathematics education researchers emphasize the beginning of this process and see research as "disciplined noticing;" they might point out new phenomena that lead to new theoretical frameworks. Others emphasize the end of the process; using theoretical frameworks, they formulate hypotheses and test them, perhaps with teaching experiments. Furthermore, many researchers today are eclectic, adjusting their approach to the particular situation they are studying.

A previous ICMI Study was devoted to the question, "What is Research in Mathematics Education and What are Its Results?" A diversity of answers emerged (Sierpinska & Kilpatrick, 1998). Perhaps this is because there are diverse ways to view learning, e.g., as individually building ideas, as the result of social interaction, or as a kind of apprenticeship. Knowledge can also be viewed in several ways, e.g., as conceptual grasp or as automated procedures. Even the mind can be viewed variously as somewhat machine-like and rule-based or more like a product of manifold associations and adaptations. Such theoretical considerations influence what is studied and how data is collected, but a major portion of research reports is taken up by descriptions of what happened to whom and why.

[2] This quote was included because it speaks to novice researchers in mathematics education. A research project in education often includes considerable investment in data collection, so it is prudent to consider early whether the question(s) can really be answered satisfactorily and whether the results might be sufficiently interesting to warrant publication.

This is not to suggest that research problems in either mathematics or mathematics education are easy to solve or that one can really know before trying whether a problem is (partially) solvable. While there exist mathematicians who report no difficulty finding problems in the literature, neglecting "problem identification" may be one factor in explaining why the Ph.D., which is ostensibly a research degree, yields such a low percentage of persons who sustain their research in either mathematics or mathematics education.

[3] The terms "theory" and "theoretical framework" have been used in a variety of ways within the mathematics education research literature. While some researchers (e.g., Schoenfeld, 1998; Dubinsky & McDonald **[Note to editors: Also put a cross-reference to the Dubinsky & McDonald paper in these volumes.]**) have delineated their ideas of what constitutes a theory (including that it support prediction, have explanatory power, and be applicable to broad ranges of phenomena), there appears to be no consistent usage within the field of mathematics education research. Thus, in this chapter, we have endeavored to use the authors' own designations, e.g., APOS theory, theory of didactical situations, local instructional theories, etc. We do not see this as the proper place to discuss the philosophical question of what constitutes a theory.

[4] Cobb (1994) describes the constructivist and sociocultural views as complementary with the former using terms like accommodation, and the latter, terms like appropriation.

[5] The idea of epistemological obstacle was introduced by G. Bachelard, and subsequently Brousseau (1983) imported this idea into mathematics education research via his theory of didactical situations. B. Cornu, and somewhat later A. Sierpinska (1985), analyzed epistemological obstacles in the development of the limit concept and linked these to pupils' behavior.

[6] APOS theory extends some of Piaget's work and use a number of the same concepts, e.g., interiorization, encapsulation, reflective abstraction, etc.

[7] While this distinction is surely in the literature somewhere, our first acquaintance with these useful terms came from a plenary lecture given by Anna Sierpinska at the 1996 Research in Undergraduate Mathematics Education Conference in Mount Pleasant, Michigan. This conference was briefly described in Selden & Selden (1997). Barbara Edwards made a similar distinction in her Pennsylvania State University Ph.D. dissertation, using the terms "logical" vs. "lexical" (for synthetic vs. analytic), which she imported from lexicography.

[8] The national impact of this ten-year calculus reform movement has been evaluated in a specially commissioned study (Ganter, in press).

[9] ISETL (Interactive SET Language) is a computer language designed to implement many mathematical constructions in a manner very close to that of ordinary mathematics (Appendix to Chapter 14, Tall, 1991). For example, finite sets can be listed in the usual way within braces, {}.

[10] See, for example, the Dubinsky and Leron (1994) textbook for abstract algebra; the course and its rationale have been described in an article for mathematicians (Leron & Dubinsky, 1995).

[11] The theory of didactical transposition proposes certain laws (regularities) with which to analyze scholarly knowledge (mathematics) in order to transform it into a form that can be taught to students (Chevallard, 1991).

Some of Brousseau's work on the theory of didactical situations, originally published in French, has recently been translated into English (Brousseau, 1997). Brousseau himself worked primarily at the school level but other French researchers have applied and extended his ideas to the university level.

[12] Using this differential equations course as a generic example, Artigue (1994) has illustrated the phases one goes through in didactical engineering. For example, prior to the design of the course, there was an analysis of the epistemological aims of the project and a thorough analysis of the constraints (e.g., the long dominance in history of the algebraic approach to differential equations and the late emergence of the geometric approach, the increased cognitive difficulty for students of having to move between algebraic, geometric, and numerical ways of thinking, etc.).

[13] However, Moore's students have been interviewed regarding his teaching method (Forbes, 1980) and the method itself has been described by Jones (1977), a mathematician who used it extensively.

[14] The term "discovery learning" has only recently been applied to Moore method teaching. Presumably it refers to the fact that students discover the proofs, and occasionally, some of the theorems. It does not appear to be derived from the discovery learning of the 1960s, espoused by Harvard psychologist J. S. Bruner and others.

[15] Here we are using terminology common in the U.S. By contrast, in the U.K. a "course" is the entire "programme" that a student takes to earn a university degree; this, in turn, consists of "modules" or "units." One of these units is referred to as a course in the U.S.

[16] *UME Trends: News and Reports on Undergraduate Mathematics Education* was published from 1989-1996, initially with support from the National Science Foundation.

[17] There is currently such a list under "Readings" in the article "Research on Undergraduate Mathematics Education: A Way to Get Started" [http://www.maa.org/data/features/rumec.html]; however it is not now regularly updated.

[18] See, for example, the research review by Leinhardt, et al (1990) or the MAA Notes Volume edited by Harel & Dubinsky (1992). More recent research by Carlson (1998) showed that many of the best students at a major U.S. state university did not acquire a fully versatile grasp of the function concept until their first year of graduate school.

[19] For example, undergraduate students might learn about mathematical proofs by discussing specific proofs many of whose parts (e.g., symbols, technical terms, statements, etc.) were linked through hypertext to explanatory information important for understanding. For example, a statement in a given proof might be linked to questions that should be asked during its validation. This, in turn, might be linked to information about logic, sets, definitions, etc. If the statement resulted from using the contrapositive of a known theorem, there might be a link to an explanation of the contrapositive which, in turn, might be linked to an explanation of the conditional (and related logical information).

[20] PME, the International Group for the Psychology of Mathematics Education, is a subgroup of ICMI, the International Commission on Mathematics Instruction. PME was established in 1976 at ICME3 in Karlsruhe. Both PME and PME-NA, its North American Chapter, have annual conferences for mathematics education researchers and publish only refereed papers in their *Proceedings*.

[2121] Future interests of Working Group members include: the secondary/tertiary interface, beliefs-attitudes-emotions of entering university students, quantification, infinity, the role of mathematical modeling, the effect of errors in textbooks on students' misconceptions, non-routine tasks, formalism, proof, use of technology in differential equations, students' understanding of the phase plane, Web course construction, comparison of analysis courses, assessment, evaluation of courses, a longitudinal study of computer use, gender issues vis-a-vis mathematics and computers, views of mathematicians on the nature of mathematics.

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