DEPARTMENT OF MATHEMATICS

TECHNICAL REPORT

ON THE DECOMPOSITION OF CLIFFORD ALGEBRAS OF ARBITRARY BILINEAR FORM

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November 1999

No. 1999-10

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On the Decomposition of Clifford Algebras of Arbitrary Bilinear Form*

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September, 27 1999

Abstract

Clifford algebras are naturally associated with quadratic forms. These algebras are $\mathbb{Z}_2$-graded by construction. However, only a $\mathbb{Z}_n$-gradation induced by a choice of a basis, or even better, by a Chevalley vector space isomorphism $C\ell(V) \leftrightarrow \bigwedge V$ and an ordering, guarantees a multi-vector decomposition into scalars, vectors, tensors, and so on, mandatory in physics. We show that the Chevalley isomorphism theorem cannot be generalized to algebras if the $\mathbb{Z}_n$-grading or other structures are added, e.g., a linear form. We work with pairs consisting of a Clifford algebra and a linear form or a $\mathbb{Z}_n$-grading which we now call *Clifford algebras of multi-vectors* or *quantum Clifford algebras*. It turns out, that in this sense, all multi-vector Clifford algebras of the same quadratic but different bilinear forms are non-isomorphic. The usefulness of such algebras in quantum field theory and superconductivity was shown elsewhere. Allowing for arbitrary bilinear forms however spoils their diagonalizability which has a considerable effect on the tensor decomposition of the Clifford algebras governed by the periodicity theorems, including the Atiyah-Bott-Shapiro mod 8 periodicity. We consider real algebras $C\ell_{p,q}$ which can be decomposed in the symmetric case into a tensor product $C\ell_{p-1,q-1} \otimes C\ell_{1,1}$. The general case used in quantum field theory lacks this feature. Theories

*Paper presented at the 5th International Conference on Clifford Algebras and their Applications in Mathematical Physics, Ixtapa, Mexico, June 27 - July 4, 1999.

with non-symmetric bilinear forms are however needed in the analysis of multi-particle states in interacting theories. A connection to q-deformed structures through nontrivial vacuum states in quantum theories is outlined.
MSCS: 15A66; 17B37; 81R25; 81R50
Keywords: Clifford algebras of multi-vectors, Clifford map, quantum Clifford algebras, periodicity theorems, index theorems, spinors, spin-tensors, Chevalley map, quadratic forms, bilinear forms, deformed tensor products, multi-particle geometric algebra, multi-particle states, compositeness, inequivalent vacua

1 Why study Clifford algebras of an arbitrary bilinear form?

1.1 Notation, basics and naming

1.1.1 Notation

To fix our notation, we want to give some preliminary material. If nothing is said about the *ring* linear spaces or algebras are build over, we denote it by $\mathbf{R}$ and assume usually that it is unital, commutative and not of characteristic 2. In some cases we specialize our base ring to the field of real or complex numbers denoted as R and C.

A *quadratic form* is a map $Q : V \mapsto \mathbf{R}$ with the following properties ($\alpha \in \mathbf{R}$, $\mathbf{V} \in V$)

$$\begin{aligned} i) \qquad Q(\alpha\mathbf{V}) &= \alpha^2 Q(\mathbf{V}), \\ ii) \qquad 2g(\mathbf{x},\mathbf{y}) &= Q(\mathbf{x}-\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}), \end{aligned} \tag{1}$$

where $g(\mathbf{x},\mathbf{y})$ is bilinear and necessarily symmetric. $g(\mathbf{x},\mathbf{y})$ is called *polar bilinear form* of Q. Transposition is defined as $g(\mathbf{x},\mathbf{y})^T = g(\mathbf{y},\mathbf{x})$. Quadratic forms over the reals can always be diagonalized by a *choice* of a basis. That is, in every equivalence class of a representation there is a diagonal representative.

We consider a *quadratic space* $\mathcal{H} = (V, Q)$ as a pair of a linear space V –over the ring $\mathbf{R}$– and a quadratic form Q. This is extended to a *reflexive space* $\mathcal{H}' = (V, B)$ viewed as a pair of a linear space V and an arbitrary non-degenerate bilinear form $B = g + A$, where $g = g^T$ and $A = -A^T$ are the symmetric and antisymmetric parts respectively. g is connected to a certain Q.

We denote the finite additive group of n elements under addition modulo n as Z_n. This should not be confused with the ring Z_n also denoted the same way.

Algebras or modules can be graded by an Abelian group. If the linear space W –not the same as V–, of an algebra can be divided into a direct sum $W = W_0 + W_1 + \ldots + W_{n-1}$ and if the algebra product maps these spaces in a compatible way one onto another, see examples, so that the index labels behave like an Abelian group, one refers to a *grading* [8].

Example 1: $W = W_0 + W_1$ and $W_0W_0 \subseteq W_0$, $W_0W_1 \simeq W_1W_0 \subseteq W_1$ and $W_1W_1 \subseteq W_0$. The indices are added modulo 2 and form a group Z_2. If $W = W_0 + W_1 + \ldots + W_{n-1}$ one has e.g. $W_iW_j \subseteq W_{i+j \bmod n}$ which is a Z_n-grading.

In the case of Z_n-grading, elements of W_m are called m-vectors or homogenous multi-vectors. The elements of $W_0 \simeq \mathbf{R}$ are also called *scalars* and the elements of W_1 are *vectors*. When the Z_2-grading is considered, one speaks about even and odd elements collected in W_0 and W_1 respectively.

However, observe that the Clifford product is *not* graded in this way since with $V \simeq W_1$ and $\mathbf{R} \simeq W_0$ one has $V \times V = \mathbf{R} + W_2$ which is not group-like. Only the even/odd grading, sometimes called parity grading, is preserved, $C\ell_+C\ell_+ \subseteq C\ell_+$, $C\ell_+C\ell_- \simeq C\ell_-C\ell_+ \subseteq C\ell_-$ and $C\ell_-C\ell_- \subseteq C\ell_+$. Hence, $C\ell$ is Z_2 graded and $C\ell \simeq C\ell_+ + C\ell_- \simeq W_0 + W_1$.

Clifford algebras are displayed as follows: $C\ell(B, V)$ is a *quantum Clifford algebra*, $C\ell(Q, V)$ is a basis-free Clifford algebra, $C\ell(g, V)$ is a Clifford algebra with a choice of a basis, $C\ell_{p,q}$ and $C\ell_n$ are real and complex Clifford algebras of symmetric bilinear forms with signature p, q or of complex dimension n respectively.

1.1.2 Basic constructions of Clifford algebras

Constructions of Clifford algebras can be found at various places in literature. We give only notation and refer the Reader to these publications [6, 8, 10, 12, 14, 34, 48, 60].

Functorial: The main advantage of the tensor algebra method is its formal strength. Existence and uniqueness theorems are most easily obtained in this language. Mathematicians derive almost all algebras from the tensor algebra –the real mother of algebras– by a process called factorization. If one singles out a two-sided ideal $\mathcal{I}$ of the tensor algebra, one can calculate 'modulo' this ideal. That is all elements in the ideal are collected to form a class called 'zero' $[0] \simeq \mathcal{I}$. Every element is contained in an equivalence class due to this construction. Denote the tensor algebra as $T(V) = \mathbf{R} \oplus V \oplus \ldots \otimes_n V \oplus \ldots$ and let $\mathbf{x}, \mathbf{y}, \ldots \in V$ and $L, M, \ldots \in T(V)$. This algebra is by construction naturally Z_∞-graded for any dimension of V.

In the case of Clifford algebras, one selects an ideal of the form

$$\mathcal{I}_{C\ell} = \{X \mid X = L \otimes (\mathbf{x} \otimes \mathbf{x} - Q(\mathbf{x})\mathbf{1}) \otimes M\} \tag{2}$$

which implements essentially the 'square law' of Clifford algebras. Note, that elements of different tensor grades –scalar and grade two– are identified. Hence this ideal is not grade-preserving and the factor algebra –the Clifford algebra– cannot be Z_∞-graded (finiteness of $C\ell(V)$); and not even multi-vector or Z_n-graded with $n = \dim V$ because all indices are now mod 2. However, the ideal $\mathcal{I}_{C\ell}$ is Z_2-graded, that is, it preserves the evenness and the oddness of the tensor elements. One defines now the Clifford algebra as:

$$C\ell(Q,V) := \frac{T(V)}{\mathcal{I}_{C\ell}}. \tag{3}$$

It is clear from the construction that a Clifford algebra is unital and associative, a heritage from the tensor algebra.

Generators and relations: Physicists and most people working in Clifford analysis prefer another construction of Clifford algebras by generators and relations [17]. One chooses a set of *generators* $\mathbf{e}_i$, images of some arbitrary basis elements $\mathbf{x}_i$ of V under the usual Clifford map $\gamma : V \mapsto C\ell(V)$ in the Clifford algebra $C\ell(V)$, and asserts the validity, in the case of $\mathbf{R} = \mathsf{R}$ or C, of the normalized, 'square law':

$$\mathbf{e}_i^2 = \pm\mathbf{1}. \tag{4}$$

Using the linearity, that is polarizing this equations by $\mathbf{e}_i \mapsto \mathbf{e}_i + \mathbf{e}_j$, one obtains the usual set of relations which have to be used to 'canonify' the algebraic expressions:

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2g(\mathbf{e}_i,\mathbf{e}_j)\mathbf{1} = 2g_{ij}\mathbf{1}. \tag{5}$$

The definition of the Clifford algebra reads:

$$C\ell(g_{ij},V) \simeq \mathbf{Alg}(\mathbf{e}_i) \bmod \mathbf{e}_i\mathbf{e}_j \ = \ 2g_{ij}\mathbf{1} - \mathbf{e}_j\mathbf{e}_i\,. \tag{6}$$

While the –image of the– numbers of the base field are called *scalars*, the $\mathbf{e}_i$ and their linear combinations are called *vectors*. The entire algebra is constructed by multiplying and linear-combining the generators $\mathbf{e}_i$ modulo the relation (5). This 'modulo relation' is in fact nothing else as a 'cancellation law' which provides one with a *unique* representative of the class of tensor elements. A basis of the linear space underlying the Clifford algebra is given by *reduced monomials* in the generators, where a certain ordering has to be chosen in the index set, e.g. ascending indices or antisymmetry. A monomial build out of n generators and the linear span of such monomials is called a homogenous n-vector. Thereby a unique Z_n-grading is introduced by the choice of a basis and an ordering.

This method has the advantage of being plain in construction, easy to remember, and powerful in computational means.

1.1.3 Naming

A very important and delicate point in mathematics and physics is the appropriate naming of objects and structures. Since we deal with a very well known structure, but want to highlight special novel features, we have to give distinguishing names to different albeit well known objects, which otherwise could not be properly addressed. This section shall establish such a coherent naming, at least for this article.

Clifford algebra is often denoted, following Clifford himself and Hestenes, as 'Geometric Algebra', GA or 'Clifford Geometric Algebra' CGA or 'Clifford Grassmann Geometric Algebra' CGGA [58]. Having the advantage of being descriptive this notation has, however, also a peculiar tendency to call upon connotations and intuitions which might *not in all cases* be appropriate. Even at this stage, one has to distinguish 'Metric Geometric Algebra' MGA and 'Projective Geometric Algebra' PGA which relies on the identification of the homogenous multi-vector objects and geometrical entities [42]. In the former case, 'vectors' are identified with 'places' –position vectors– of pseudo-Euclidean or unitary spaces while in the second case 'vectors' are identified with 'points' of a projective space.

Both variants, metric or projective, use unquestionably the artificial multi-vector structure introduced by the mere notation of a basis and foreign to Clifford algebras to assert 'ontological' statements such as: '$\mathbf{x}$ is a place in Euclidean space' or '$\mathbf{x}$ is a point in a projective space'.

Both of these interpretations have one thing in common, namely, they assert an *object character* to the Clifford elements themselves. We will coin for this case the term '**Classical Clifford Algebra**'.

To our current experience, the Wick isomorphism developed below guarantees that such interpretation of Clifford algebras is *independent* from the chosen Z_n-grading. That is, we make the following *conjecture:* if the Clifford elements themselves are 'ontologically' interpreted as 'place' or 'point' then all Z_n-gradings are isomorphic through the Wick isomorphism.

We turn to the second aspect. In [56] Oziewicz introduced the term 'Clifford algebras of multi-vectors' to highlight the fact that he considered different Z_n-gradings or, equivalently, different multi-vector structures. However, Clifford algebras have in nearly every case been used as multi-vector Clifford algebras since mathematicians and physicists want to consider the n-vectors or multi-vectors for different purposes.

Following the introduction of Clifford algebras of arbitrary bilinear forms, implicitly in [12] and explicitly in [1, 24, 25, 26, 27, 28, 29, 31, 32, 49, 55], situations have occurred for good physical reasons where different Z_n-gradings have led to different physical outcomes. In those situations a theory of gradings is mandatory.

A new point is the *operational approach* to Clifford elements. If one considers a Clifford number to be an operator, it has to act on another object, a 'state vector'. This '*quantum point of view*' moves also the ontological assertions into the states. Their interpretation however is difficult.

Moreover, one has to deal with representation theory which was not necessary in the 'classical' Clifford algebraic approach –in both senses of classical, i.e. also as opposed to quantum, here. Adopting Wigner's definition of a particle as an irreducible representation –of the Poincaré group– one has to seek irreducible representations of Clifford algebras. It is a well known fact that these representations are faithfully realized in spinor spaces. It is exactly at this place where it will be shown in this article that one obtains different Z_n-gradings or different multi-vector structures leading to different results. In fact we are able to find *irreducible* spinor spaces of dimension 8 in $C\ell_{2,2}(B,V)$, where 2, 2 denotes the signature of the symmetric part g of B, and not of dimension 4 as predicted by the 'classical' Clifford algebra theory.

For the case of Clifford algebras of multi-vectors we coin the term **quantum Clifford algebra.**[1,2] It is clear to us that we risk creating a confusion with this term, which looks like a q-deformed version of an ordinary Clifford algebra, while also in our case the common 'square law' is fully valid! However, this link is not wrong! As we show elsewhere in these proceedings [4], one is able to find Hecke algebras and q-symmetry *within* the structure of the quantum Clifford algebra. It is also in accord with the attempt of G. Fiore, presented at this conference, to describe q-deformed algebras in terms of undeformed generators. This is just a reverse of our argument. However, the characteristic point in our consideration is that we dismiss the classical ontological interpretation in favor of an operational interpretation. Thereby it is necessary to study states which are now Z_n-grade dependent. Our approach should be contrasted by the recent developments excellently described in [15, 51]. A different treatment of Clifford algebras in connection with Hecke algebras was given in [57].

As a last point, we emphasize that indecomposable spinor representations of unconventionally large dimensions are expected to be spinors of bound systems, see [27]. Hence, studying decomposability is the first step towards an algebraic theory of compositeness including stability of bound states.

1.2 Why study $C\ell(B,V)$ and not $C\ell(Q,V)$? – Physics

Clifford algebras play without any doubt a predominant role in physics and mathematics. This fact was clearly addressed and put forward by D. Hestenes [38, 39, 40, 41]. Based on this solid ground, we give an analysis of Clifford algebras of an arbitrary bilinear form which exhibit novel features especially regarding their representation theory. The most distinguishing fact between our approach and usual treatments of Clifford algebras e.g., [6, 10, 14, 48, 60], is that we seriously consider how the Z_n-grading is introduced in Clifford algebras. This is most important since Clifford algebras are *only* Z_2-graded by their natural –functorial– construction. The introduction of a further finer grading does therefore put new assumptions into the theory. One might therefore ask, if theses additional structures are important or even necessary in physics and mathematics.

Indeed, after examining various cases we notice that *every* application of Clifford algebras which is computational –not only functorial– deals in fact with the so called Clifford algebras of multi-vectors [56] or *quantum Clifford algebras*. However, the additional Z_n-grading, even if mathematically and physically necessary for applications, is usually introduced without any ado. Looking at literature we can however find lots of places where Z_n-graded Clifford algebras are

[1]This is close to Saller's notion of a "quantum algebra" which denotes however a special choice of grading [61].

[2]Classical Clifford algebras emerge as a particular case of quantum Clifford algebras.

not only appropriate but needed. This is in general evident in every quantum mechanical setup.

If one analyzes functional hierarchy equations of quantum field theory (QFT), one is able to translate these functionals with a help of Clifford algebras. Such attempts have already been made by Caianiello [11]. He noticed that at least two types of orderings are needed in QFT, namely the time-ordering and normal-ordering. Since one has –at least– two possibilities to decompose Clifford algebras into basis monomials, he introduces Clifford and Grassmann bases. A basis of a Clifford algebra is usually given by monomials with totally ordered index sets. If one has a finite number of 'vector' elements $\mathbf{e}_i$, one can, by using the anti-commutation relations of the Clifford algebra, introduce the following bases

$$\begin{array}{ll} i) & \{\mathbf{1};\mathbf{e}_1,\ldots,\mathbf{e}_n;\mathbf{e}_1\mathbf{e}_2,\ldots;\mathbf{e}_{i_1}\mathbf{e}_{i_2}\mathbf{e}_{i_3\,(i_1<i_2<i_3)},\ldots\} \\ ii) & \{\mathbf{1};\mathbf{e}_1,\ldots,\mathbf{e}_n;\mathbf{e}_{[1}\mathbf{e}_{2]},\ldots;\mathbf{e}_{[i_1}\mathbf{e}_{i_2}\mathbf{e}_{i_3]},\ldots\}. \end{array} \tag{7}$$

We used the $[\ldots]$ bracket to indicate antisymmetrization in the index set. An ordering of index sets is inevitable since the $\mathbf{e}_i\mathbf{e}_j$ and $\mathbf{e}_j\mathbf{e}_i$ monomials are not algebraically independent due to the anti-commutation relations $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i + g_{ij}\mathbf{1}$. Caianiello identifies then the two above choices with time- and normal-ordering. However, already at this point it is questionable why one uses 'lexicographical' ordering '$<$' and not e.g. the 'anti-lexicographical' ordering '$>$' or an ordering which results from a permutation of the index set.

A detailed study shows that fermionic QFT needs antisymmetric index sets and that there are infinitely many such choices [25, 31]. Using this fact we have been able to show that singularities, which arise usually due to the reordering procedures such as the normal-ordering, are no longer present in such algebras [32]. Studying the transition from operator dynamics to functional hierarchies, the so-called Schwinger-Dyson-Freese hierarchies, in [25, 31] it turned out that the multi-vector structure, or, equivalently a uniquely chosen Z_n-grading, was a *necessary input* to QFT.

Multi-particle systems provide a further place where a careful study of gradings will be of great importance. It is a well known fact that one has the Clebsch-Gordan decomposition of two spin-$\frac{1}{2}$ particles as follows [33, 37]:

$$\frac{\mathbf{1}}{\mathbf{2}} \otimes \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0} \oplus \mathbf{1}\,. \tag{8}$$

However, since this is an identity, it can be used either from left to right to form bosonic spin 0 and spin 1 'composites' *or* from right to left! There is no way –besides the experience– to distinguish if such a system is composed, that is, dynamically stable or not, see [26]. From a mathematical point of view one cannot distinguish n free particles from an n-particle bound system by means of algebraic considerations. This is seen clearly in the decomposition theorems for Clifford algebras where larger Clifford algebras are decomposed into smaller blocks of Clifford tensor factors. This cannot be true for bound objects which lose their physical character when being decomposed. An electron and proton system is quite different from a hydrogen atom. In this work, we will see, that one can indeed find such *indecomposable* states in quantum Clifford algebras.

This raises a question how to distinguish such situations. One knows from QFT that interacting systems have to be described in non-Fock states and that there are infinitely many such representations [35]. It is thus necessary to introduce the concept of *inequivalent states* in finite dimensional systems [27, 44, 45]. Such states are necessarily non-Fock states, since Fock states belong to systems of non-interacting particles. This is the so-called *free case* which is however very useful in perturbation theory. The present paper supports the situation found in [27].

Closely related to these inequivalent states are *condensation phenomena*. As it was shown in [27], one can algebraically determine boundedness using an appropriate Z_n-grading. Furthermore, it was shown that the dynamics *determines* correct grading. In BCS theory of superconductivity the fact that bound states can or cannot be build was shown to imply a gap-equation [27] which governs the phase transition.

A further point related to Z_n-graded Clifford algebras is q-quantization. This can be seen when studying physical systems as in [30] and when adopting a more mathematical point of view as in [24, 28]. In these proceedings a detailed example was worked out to show how q-symmetry and Hecke algebras can be described within quantum Clifford algebras [4]. It is quite clear that this structure should play a major role in the discussion of the Yang-Baxter equation, the knot theory, the link invariants and in other related fields which are crucial for the physics of integrable systems in statistical physics.

However, the most important implication from these various applications is that the q-symmetry and more general deformations are *symmetries of composites*. This was already addressed in [30] and more recently in [24]. Also the present work provides full support for this interpretation, as the talk of G. Fiore at this conference. Providing as much evidence as possible to this fact was a major motivation for the present work.

1.3 Why study $C\ell(B,V)$ and not $C\ell(Q,V)$? – Mathematics

There are also arguments of purely mathematical character which force us to consider quantum Clifford algebras.

If we look at the construction of Clifford algebras by means of the tensor algebra, we notice that $C\ell$ is a functor. To every quadratic space $\mathcal{H} = (V, Q)$, a pair of a linear space V over a ring $\mathbf{R}$ and a quadratic form Q, there is a uniquely connected Clifford algebra $C\ell(Q, V)$. That is, one can introduce the algebra structure without any further input or choices, so to say for free. One may further note that if the characteristic of the ring $\mathbf{R}$ is not 2, then there is a one-to-one correspondence between quadratic forms and classes of symmetric matrices [62]. In other words, every symmetric matrix is a representation of a quadratic form in a special basis. Over the reals (complex numbers) the classes of quadratic forms can be labeled by dimension n and signature s (dimension n only, no signature in C). Equivalently one can use the numbers p, q of positive and negative eigenvalues of the quadratic form. This leads to a classification (naming) of real (and complex) Clifford algebras. One writes $C\ell(Q,V) \simeq C\ell_{p,q}$ $(C\ell_n)$ where dim $V = n = p+q$ and Q has signature $s = p-q$. The remarkable fact is that the 'square law' for vectors $Q(\mathbf{v}) \equiv \mathbf{v}^2 = \alpha\mathbf{1} \in C\ell(Q;V)$ $(\alpha \in \mathsf{R}$ or $\alpha \in \mathsf{C})$ is a diagonal map determining only the symmetric part of the map $Q(\mathbf{V}) \mapsto \mathsf{R}$. Following Clifford one should note that the product operation can be seen as acting on the second factor $2 \times x$ as a doubling of x; that is, $2\times$ is a doubling operator or endomorphism acting on the space of the second factor. In this sense any 'Clifford number' induces an endomorphism on the graded space W underlying the algebra and it is questionable why one should use only diagonal maps and their symmetric polarizations. Furthermore, note that one has

$$\text{quadratic forms} \quad \simeq \quad \frac{\text{bilinear forms}}{\text{alternating forms}}. \tag{9}$$

The dualization $V \mapsto V^* \simeq \text{lin-Hom}(V,\mathsf{R})$ is performed by an arbitrary (non-degenerate) bilinear form. Endomorphisms have in general the following form

$$\text{End}(V) \simeq V \otimes V^*, \tag{10}$$

so why do we restrict ourselves to the symmetric case? If we consider a pair (V,B) of a space V and an arbitrary bilinear form B, can we construct functorially an algebra like the Clifford algebra for the pair $\mathcal{H} = (V,Q)$?

It can be easily checked that if one insists on the validity of the 'square law' $\mathbf{v}^2 = \alpha\mathbf{1}$, the *anti-commutation relations* of the resulting algebra are the same as for usual Clifford algebras while the *commutation relations* –and thus the meaning of ordering and grade– is changed. Let $B = g+A$, $A^T = -A$, $g^T = g$. We denote $B(\mathbf{x},\mathbf{y}) = \mathbf{x} \lrcorner_B \mathbf{y}$, $A(\mathbf{x},\mathbf{y}) = \mathbf{x} \lrcorner_A \mathbf{y}$ and $g(\mathbf{x},\mathbf{y}) = \mathbf{x} \lrcorner_g \mathbf{y}$ (the latter also denoted by Hestenes and Sobczyk as $\mathbf{x}\cdot\mathbf{y}$).[3] Then, the B-dependent Clifford product $\underset{B}{\mathbf{x}\mathbf{y}}$ of two 1-vectors $\mathbf{x}$ and $\mathbf{y}$ in $C\ell(B,V)$ can be decomposed in *different ways* into scalar and bi-vector parts as follows

$$\begin{aligned}
\underset{B}{\mathbf{x}\mathbf{y}} &= \mathbf{x} \underset{g}{\lrcorner} \mathbf{y} + \mathbf{x} \,\dot{\wedge}\, \mathbf{y} && \text{Hestenes, common case, } A = 0 \\
\underset{B}{\mathbf{x}\mathbf{y}} &= \mathbf{x} \underset{B}{\lrcorner} \mathbf{y} + \mathbf{x} \wedge \mathbf{y} && \text{Oziewicz, Lounesto, Abłamowicz, Fauser,}
\end{aligned} \tag{11}$$

where $\mathbf{x} \,\dot{\wedge}\, \mathbf{y} = \mathbf{x} \wedge \mathbf{y} + A(\mathbf{x},\mathbf{y}) = \mathbf{x} \wedge \mathbf{y} + \mathbf{x} \lrcorner_A \mathbf{y}$. Of course, for any 1-vector $\mathbf{x}$ and any element u in $C\ell(B,V)$ we have:

$$\underset{B}{\mathbf{x}u} = \mathbf{x} \underset{B}{\lrcorner} u + \mathbf{x} \wedge u = \mathbf{x} \underset{g}{\lrcorner} u + \mathbf{x} \underset{A}{\lrcorner} u + \mathbf{x} \wedge u = \mathbf{x} \underset{g}{\lrcorner} u + \mathbf{x} \,\dot{\wedge}\, u. \tag{12}$$

Notice that the element $\mathbf{x} \,\dot{\wedge}\, u = \mathbf{x} \lrcorner_A u + \mathbf{x} \wedge u$ is not even a homogenous multi-vector in $\bigwedge V$. We have thus established that the multi-vector structure

[3] The symbols $\lrcorner_B$, $\lrcorner_A$ and $\lrcorner_g$ denote the left contraction in C`(B,V) with respect to B, A and g respectively.

is uniquely connected with the antisymmetric part A of the bilinear form, see also [1, 29, 31].

This has an immediate consequence: in some cases one finds bi-vector elements which satisfy minimal polynomial equations of the Hecke type [24, 28]. This feature is treated extensively elsewhere in this Volume [4].

Some mathematical formalisms, not treated here, are closely connected to this structure. One is the structure theory of Clifford algebras over arbitrary rings [36] where a classification is still lacking. Connected to these questions is the arithmetic theory of Arf invariants and the Brauer-Wall groups.

Much more surprising is the fact that due to central extensions the ungraded bi-vector Lie algebras turn into Kac-Moody and Virasoro algebras [54] and, as it is also shown in [4], to some q-deformed algebras.

Since Clifford algebras naturally contain reflections, automorphisms generated by non-isotropic vectors, we expect to find infinite dimensional Coxeter groups [17, 43], affine Weyl groups etc., connected to Z_n-graded or quantum Clifford algebras.

Involutions connected to special elements, norms and traces [36] are also affected by different gradings. This has considerable effects. One important point is that the Cauchy-Riemann differential equations are altered which makes probably the concept of monogeneity [48] grade dependent. However, this is speculative.

2 Chevalley's approach to Clifford algebras

2.1 Confusion with Chevalley's approach

Chevalley's book *"The algebraic theory of spinors"* [12] seems to have been badly accepted by working mathematicians and physicists despite its frequent citation. Albert Crumeyrolle stated the following in [14], p. xi:

> *In spite of its depth and rigor, Chevalley's book proved too abstract for most physicists and the notions explained in it have not been applied much until recently, which is a pity.*

The more compact and readable book *"The study of certain important algebras"* [13] seems to be little known. However, one can find in many physical writings e.g. Berezin [7] very analogous structures, without mentioning the much more complete work of Chevalley.

When looking for the most general construction of Clifford algebras over arbitrary rings including the case where the characteristic of $\mathbf{R}$ is 2, Chevalley constructed the so-called *Clifford map.* This map is an injection of the linear space V into the algebra $C\ell(V)$ which establishes the 'square law'. This construction emphasizes the operator character of Clifford algebras and establishes a connection between the spaces underlying the Z_n-graded Grassmann algebra and the thereon constructed Clifford algebra. For our purpose it is important that *only* Chevalley's construction allows a non-symmetric bilinear form in constructing Clifford algebras. However, this fact is not explicit in Chevalley's writings but it is clearly emphasized in [55].

Ironically, a careful analysis of Lounesto shows that even Crumeyrolle made a mistake in describing the Chevalley isomorphism connecting Grassmann and Clifford algebra spaces. In [50] Lounesto points out that Crumeyrolle rejects the Chevalley isomorphism for *any* characteristic. This seems to be implied by Crumeyrolle's frequent questioning, also in previous Clifford conferences of this series: "What is a bi-vector?" [53]. However, an isomorphism can be uniquely given if the characteristic of $\mathbf{R}$ is not 2, see [49, 50]. On the other hand, Lounesto points out that Lawson and Michelsohn [47] postulate such an isomorphism which is wrong in the exceptional case of characteristic 2. One should note in this context that their point of view is taken by almost all working mathematicians and physicists.

At this point we submit, that we insist on Chevalley's construction even in the case of characteristic not 2. Lounesto claims that in this cases $C\ell(B,V)$ is isomorphic to $C\ell(Q,V)$ with Q the quadratic form associated to B. In fact, this is true for the Clifford algebraic structure and was proved in [1] up to the dimension 9 of V. However, this, the so-called *Wick isomorphism* between $C\ell(B,V)$ and $C\ell(Q,V)$, has to be rejected when the Z_n-grading is considered, or, in other words, the multi-vector structure. Hence, we reject Lounesto's judgment that it is worth studying $C\ell(B,V)$ only in characteristic 2 for the reason of carefully treating the involved Z_n-grading or multi-vector structure. This is one of the main points of our analysis.

2.2 Chevalley's construction of $C\ell(B,V)$

A detailed and mathematical rigorous development of quantum Clifford algebras $C\ell(B,V)$ can be found in [24, 31]. We will develop only the notation and point out some peculiar features insofar as they appear in the present study, see also [1, 29].

The main feature of the Chevalley approach is that Clifford algebras are constructed as special –satisfying the 'square law'– endomorphism algebras on –the linear space of– a Grassmann algebra. In this way the Grassmann algebra, which is naturally Z_n-graded, induces via the Chevalley isomorphism a grading or multi-vector structure in the Clifford algebra. This grading is however not preserved by the Clifford product which renders the Clifford algebra to be a deformation of the Grassmann algebra.

To proceed along this line we construct the Grassmann algebra as a factor algebra of the tensor algebra. Let

$$\mathcal{I}_G := \{X \mid X = A \otimes (\mathbf{x} \otimes \mathbf{x}) \otimes B\} \tag{13}$$

with notation as in (2) and define

$$\bigwedge V := \frac{T(V)}{\mathcal{I}_G}, \quad \pi : T(V) \mapsto \bigwedge V\,. \tag{14}$$

The projected tensor product $\pi(\otimes) \mapsto \wedge$ is denoted as wedge or outer product. The induced grading is

$$\bigwedge V = \mathbf{R} \oplus V \wedge V \oplus \ldots \oplus \wedge^n V \oplus \ldots \quad . \tag{15}$$

As the next step, we consider reflexive duals of the linear space V. Define

$$V^* := \text{lin-Hom}(V, \mathbf{R}) \tag{16}$$

where $\dim V^* = \dim V$ (reflexivity). Using the action of the dual elements on V we define the (left) contraction $\lrcorner_B$ as:

$$i_{\mathsf{x}}(\mathbf{y}) = \mathbf{x} \underset{B}{\lrcorner} \mathbf{y} \; = \; B(\mathbf{x}, \mathbf{y}). \tag{17}$$

Note, that $i_{\mathsf{x}} \in V^*$ is the dualized element $\mathbf{x}$ and that here a *certain* duality map is employed. If this is the usual duality map $i_{\mathsf{e}_{\mathsf{i}}}(\mathbf{e}_j) = \delta_{ij}$ one denotes this as Euclidean dual isomorphism and writes the map as $\star$ [61]. The notation $\mathbf{x} \lrcorner_B \mathbf{y}$ and much more $B(\mathbf{x}, \mathbf{y})$ is very peculiar since we have

$$\underset{B}{\lrcorner} \, : \, V \times V \mapsto V, \qquad B \, : \, V \times V \mapsto V. \tag{18}$$

Hence, $\lrcorner_B$ and B are in lin-Hom$(V \times V, \mathbf{R}) \simeq V^* \times V^*$. In this notation a dual isomorphism is *implicitly* involved, since we consider really maps of the form

$$< . \mid . >: \, V^* \times V \mapsto \mathbf{R} \tag{19}$$

which might be called *a dual product* or *a pairing* [8, 61].

Having defined the action of V^* on V, we lift this action to the entire Grassmann algebras $\bigwedge V$ and $\bigwedge V^*$. For $\mathbf{x}, \mathbf{y} \in V$, and $u, v, w \in \bigwedge V$ we have:

$$\begin{aligned}
&i) \qquad \mathbf{x} \underset{B}{\lrcorner} \mathbf{y} = B(\mathbf{x}, \mathbf{y}), \\
&ii) \qquad \mathbf{x} \underset{B}{\lrcorner} (u \wedge v) = (\mathbf{x} \underset{B}{\lrcorner} u) \wedge v + \hat{u} \wedge (\mathbf{x} \underset{B}{\lrcorner} v), \\
&iii) \qquad (u \wedge v) \underset{B}{\lrcorner} w = u \underset{B}{\lrcorner} (v \underset{B}{\lrcorner} w),
\end{aligned} \tag{20}$$

where $\hat{}$ is the involutive map –grade involution– $\hat{} : V \mapsto -V$ lifted to $\bigwedge V$. The Clifford algebra $C\ell(B, V)$ is then constructed in the following way. Define an operator $L^{\pm}_{\mathsf{x}} : \bigwedge V \mapsto \bigwedge V$ for any $\mathbf{x} \in V$ as:

$$(L^{\pm}_{\mathsf{x}})^2 := \mathbf{x} \underset{B}{\lrcorner} \cdot \, \pm \, \mathbf{x} \wedge \cdot \tag{21}$$

and observe that this is a Clifford map [12, 24, 31]

$$(L^{\pm}_{\mathsf{x}})^2 = \pm Q(\mathbf{x}) \mathbf{1}, \tag{22}$$

where $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{y})$. This is nothing else as again the 'square law', and one proceeds as in the case of generators and relations. Chevalley has thus established that

$$C\ell(B, V) \subset \text{End}(\bigwedge V). \tag{23}$$

This inclusion is strict.

3 Wick isomorphism and Z_n-grading

3.1 Wick isomorphism

In this section we will prove the following **Theorem:**

$$C\ell(B,V) \cong C\ell(Q,V) \tag{24}$$

as Z_2-graded Clifford algebras.

This isomorphism, denoted below by ϕ, is the *Wick isomorphism* since it is the well know normal-ordering transformation of the quantum field theory [21, 31, 64]. This was not noticed for a long time which is another 'missed opportunity' [22].

Proof: The proof proceeds in various steps, numbered by letters a, b, c, etc. After defining the outer exponential, we prove the following important formulas:

$$\begin{array}{rl} i) & e_\wedge^{-F} \wedge e_\wedge^{F} = \mathbf{1}, \\ ii) & e_\wedge^{-F} \wedge \mathbf{x} \wedge e_\wedge^{F} \wedge u \;=\; \mathbf{x} \wedge u, \\ iii) & e_\wedge^{-F} \wedge (\mathbf{x} \underset{g}{\rfloor} (e_\wedge^{F} \wedge u)) \;=\; \mathbf{x} \underset{g}{\rfloor} u + (\mathbf{x} \underset{g}{\rfloor} F) \wedge u, \end{array} \tag{25}$$

and finally we show that the Wick isomorphism ϕ is given as:

$$\begin{aligned} C\ell(B,V) &= \phi^{-1}(C\ell(g,V)) \\ &= e_\wedge^{-F} \wedge C\ell(Q,V) \wedge e_\wedge^{F} && (26) \\ &\cong (\,C\ell(g,V), <.>_r^A\,) && (27) \end{aligned}$$

where $<.>_r^A$ denotes the A-dependent Z_n-grading.

That is, the isomorphism is given by the following transformation of *vector* variables which is then algebraically lifted to the entire algebra:

$$\begin{array}{rcl} \mathbf{x} \underset{g}{\rfloor} \cdot & \rightarrow & \mathbf{x} \underset{B}{\rfloor} \cdot \;=\; \mathbf{x} \underset{g}{\rfloor} \cdot \,+\, (\mathbf{x} \underset{g}{\rfloor} F) \wedge \cdot \\ \mathbf{x} \wedge \cdot & \rightarrow & \mathbf{x} \wedge \cdot \end{array} \tag{28}$$

a) According to Hestenes and Sobczyk [39] it is possible to express every antisymmetric bilinear form in the following way

$$A(\mathbf{x},\mathbf{y}) := F \underset{g}{\rfloor} (\mathbf{x} \wedge \mathbf{y}) \tag{29}$$

where F is an appropriately chosen bi-vector. F can be decomposed in a non-unique way into homogenous parts $F_i = \mathbf{a}_i \wedge \mathbf{b}_i$, $F = \sum F_i$. We define the outer exponential of this bi-vector as $(\wedge^0 F = \mathbf{1})$

$$e_\wedge^{F} := \sum \frac{1}{n!} \wedge^n F \;=\; \mathbf{1} + F + \frac{1}{2} F \wedge F + \ldots + \frac{1}{n!} \wedge^n F + \ldots\;. \tag{30}$$

This series is finite when the dimension of V is finite since in that case there exists a term of the highest grade.

b) Substitute the series expansion (30) into (25-i) and note that after applying the Cauchy product formula for sums we have

$$e_{\wedge}^{-F} \wedge e_{\wedge}^{F} = \sum_{r=0}^{\infty} \left(\sum_{l=0}^{r} (-1)^l \binom{r}{l} \right) \frac{1}{r!} \wedge^r F. \tag{31}$$

The alternating sum of the binomial coefficient is zero except in the case $r = 0$ when we obtain $\mathbf{1}$, which proves formula (25-i).
c) To prove (25-iii) one needs the commutativity of $\mathbf{x} \,\lrcorner_g\, F_i$ with F_j. If the contraction is zero, it commutes trivially, if not, the contraction is a vector $\mathbf{y}$. From $\mathbf{y} \wedge F = F \wedge \mathbf{y}$ for every bi-vector, we have that $\mathbf{x} \,\lrcorner_g\, F_i$ commutes with any F_j and thus with F. This allows us to write

$$\mathbf{x} \underset{g}{\lrcorner} (\wedge^n F) = n(\mathbf{x} \underset{g}{\lrcorner} F) \wedge (\wedge^{(n-1)} F). \tag{32}$$

Once more using $\wedge^0 F = \mathbf{1}$, the Leibniz' rule and the fact that $\hat{F} = F$, we obtain

$$\mathbf{x} \underset{g}{\lrcorner} (e_{\wedge}^{F} \wedge u) = e_{\wedge}^{F} \wedge (\mathbf{x} \underset{g}{\lrcorner} u + (\mathbf{x} \underset{g}{\lrcorner} F) \wedge u), \tag{33}$$

which proves (25-iii).
d) Since any vector $\mathbf{y}$ commutes under the wedge with any bi-vector F, the case (25-ii) reduces to **b)**.
e) The Wick isomorphism is now given as $C\ell(B, V) = \phi^{-1}(C\ell(g, V)) = e_{\wedge}^{-F} \wedge C\ell(Q, V) \wedge e_{\wedge}^{F}$. The same transformation can be achieved by decomposing every Clifford 'operator' into vectorial parts and then into contraction and wedge parts w.r.t. $(g, \wedge)$ and then performing the substitution laws given in (28) and a final renaming of the contractions; see [31] for an application in quantum field theory.

Note, that since the wedges are *not* altered and the new contractions are given by $\mathbf{x} \,\lrcorner_B\, \cdot \equiv \mathrm{d}_{\mathsf{x}}(\,\cdot\,) := \mathbf{x} \,\lrcorner_g\, \cdot + (\mathbf{x} \,\lrcorner_g\, F) \wedge \cdot$, this transformation does mix grades, but it respects the parity. It is thus a Z_2-graded isomorphism. QED.

An equivalent proof was delivered in [63] without using (explicitly) Clifford algebras but index doubling –see below. The Wick isomorphism was called there 'nonperturbative normal-ordering'.

3.2 $C\ell(B, V) \leftrightarrow C\ell(Q, V)$ – Isomorphic yet different?

We have already discussed that many researchers reject the idea that $C\ell(B, V)$ is of any use because of the Wick isomorphism. However, as our proof has shown this isomorphism is only Z_2-graded. Indeed it was not the mathematical opportunity, but a necessity in modeling quantum physical multi-particle systems and quantum field theory which forced us to investigate quantum Clifford algebras [25, 27, 31, 32].

Decomposing B into g, A as in (12) and noting that in our case, of characteristic not 2 one has $Q(\mathbf{x}) = g(\mathbf{x},\mathbf{x})$, one concludes that $C\ell(Q,V)$ is exactly the equivalence class of $C\ell(B,V) \simeq C\ell(g+A,V)$ with A varying arbitrarily:

$$C\ell(Q,V) = [C\ell(g+A,V)]\,. \tag{34}$$

In other words, one does not have a single Clifford algebra $C\ell(Q,V)$ but an entire class of equivalent –under the Z_2-graded Wick isomorphism– Clifford algebras $C\ell(B,V)$. This can be written as

$$C\ell(Q,V) \simeq C\ell(g+A,V) \bmod A \tag{35}$$

which induces a unique projection from the class of quantum Clifford algebras onto the classical Clifford algebra. Such a projection π can be defined as:

$$\begin{array}{ll} i) & \pi \,:\, T(V) \mapsto C\ell(B,V) \\ ii) & <\,.\,>_r^A := \pi(\otimes^r V). \end{array} \tag{36}$$

This is once more a sort of 'cancellation law'. The important fact is that *only* those properties belong to $C\ell(Q,V)$ which do *not* depend on the particular choice of a representant parameterized by A. Physically speaking, only those properties belong to $C\ell(Q,V)$ which are homogenous over the entire equivalence class.

As we will show now, especially the multi-vector Z_n-grading is *not* of this simple type. Recall that it is possible to decompose the Clifford product in various ways as in (11) and (12). Hence we obtain a relation between the $\wedge$-and the $\dot\wedge$-grading as:

$$\mathbf{x} \,\dot\wedge\, \mathbf{y} = A(\mathbf{x},\mathbf{y}) + \mathbf{x} \wedge \mathbf{y} \tag{37}$$

which shows that a $\dot\wedge$-bi-vector is an inhomogeneous $\wedge$-multi-vector and vice versa. Since the antisymmetric part can be absorbed in the wedge product, using the Wick isomorphism, we can give the grading explicitly by writing

$$<\,.\,>_r^A \;=\; <\,.\,>_r^{\dot\wedge} \tag{38}$$

with respect to the doted wedge $\dot\wedge$ *within* the undeformed algebra $C\ell(Q,V)$, see G. Fiore's talk. This gives us a second characterization of $C\ell(B,V)$, namely

$$C\ell(B,V) \simeq (C\ell(Q,V), <\,.\,>_r^A)\,. \tag{39}$$

That is, $C\ell(B,V)$ can be seen as a pair of a classical Z_2-graded Clifford algebra $C\ell(Q,V)$ and a unique multi-vector structure given by the projectors $<\,.\,>_r^A$. As a main result we have that these algebras are *not* isomorphic under the Wick isomorphism

$$C\ell(g+A_1,V) \underset{\text{Wick}}{\not\simeq} C\ell(g+A_2,V) \quad \text{iff} \quad A_1 \neq A_2\,. \tag{40}$$

4 Periodicity theorems

Our theory will have an impact on all famous periodicity theorems of Clifford algebras, especially on the Atiyah-Bott-Shapiro mod 8 index theorem [5]. But to be as concrete and explicit as possible, we restrict ourself to the case $C\ell_{p,q} \simeq C\ell_{p-1,q-1} \otimes C\ell_{1,1}$. Periodicity theorems can be found, for example, in [6, 10, 46, 52, 60].

We need some further notation. Let $V_{p,q} = (g_{p,q}, V)$ be a quadratic space, where $g = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$ with p plus signs and q minus signs, and let V be a linear space of dimension $p+q$. According to the Witt theorem [65] one can split off a quadratic space of the hyperbolic type $M_{1,1}$. This split is orthogonal with respect to g :

$$V_{p,q} = N_{p-1,q-1} \perp_g M_{1,1} \,. \tag{41}$$

If one applies the Clifford map $\gamma : V_{p,q} \mapsto C\ell_{p,q}$ and defines its natural restrictions $\gamma' : N_{p-1,q-1} \mapsto C\ell_{p-1,q-1}$, $\gamma'' : M_{1,1} \mapsto C\ell_{1,1}$, one obtains the following **Periodicity Theorem:**

$$C\ell_{p,q} \simeq C\ell_{p-1,q-1} \otimes C\ell_{1,1}. \tag{42}$$

While in this special case the tensor product may be ungraded, in general the tensor product in such decompositions may be graded or not, see [10, 46, 52].

Using the obvious notation $C\ell(V_{p,q}) = C\ell_{p,q}(Q)$ and introducing the restrictions of the Wick isomorphism $\phi^{-1}|_N$ and $\phi^{-1}|_M$, (here $N = N_{p-1,q-1}$ and $M = M_{1,1}$), we can calculate the decomposition of $C\ell_{p,q}(B)$. However, if there are terms in the bi-vector F which connect spaces N and M, that is, if $F = \sum F_i$ and if there exists $F_s = \mathbf{a}_s \wedge \mathbf{b}_s$ with $\mathbf{a}_s \in N$, $\mathbf{b}_s \in M$, this part of the construction belongs *neither* to the restriction $\phi^{-1}|_N$ *nor* to $\phi^{-1}|_M$. We have *either* **no tensor decomposition** *or* a **deformed tensor product**. Expressed in formulas we get:

$$\begin{aligned}
C\ell_{p,q} &= \phi^{-1}(C\ell_{p,q}(Q)) \\
&= \phi^{-1}\left[C\ell_{p-1,q-1}(Q|_N) \otimes C\ell_{1,1}(Q|_M)\right] \\
&= C\ell_{p-1,q-1}(B|_N)(\phi^{-1}\otimes)C\ell_{1,1}(B|_M) \\
&= C\ell_{p-1,q-1}(B|_N) \otimes_{\phi^{-1}} C\ell_{1,1}(B|_M).
\end{aligned} \tag{43}$$

Remark: The deformed tensor product $\otimes_{\phi^{-1}}$ is not braided by construction, since we have no restrictions on ϕ^{-1}. But one is able to find e.g., Hecke elements, etc., necessary for a common q-deformation or, more generally, a braiding.

As the main result of our investigation we have shown that quantum Clifford algebras *do not* come in general with periodicity theorems as e.g. the famous Atiyah-Bott-Shapiro mod 8 index theorem. This has *enormous* impact on quantum manifold theory and the topological structure of such spaces as well as on their analytical properties. However, we have constructed a deformed –not necessarily braided– tensor product $\otimes_{\phi^{-1}}$ which gives a decomposition at the cost of losing (anti)-commutativity. To fully support this view and convince also those Readers who might consider our reasoning too abstract and only formal in nature, we proceed to provide some examples.

5 Examples

In this section we consider three examples each of them pointing out a peculiar feature of quantum Clifford algebras and Z_n-gradings. Two of these examples have been found by using CLIFFORD, a Maple V Rel. 5 package for quantum Clifford algebras [2, 3]. While the second example is generic, the third one was taken from [27] and provides an example of a physical theory which benefits extraordinarily from using quantum Clifford algebras.

5.1 Example 1

This example shows that even in classical Clifford algebras one does not have a unique access to the *objects* of the graded space. Consider the well-known Dirac γ matrices which generate the Dirac-Clifford algebra $C\ell_{1,3}$ and satisfy $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{ij}\mathbf{1}$ with the Minkowski metric $\eta_{ij} = \mathrm{diag}(1,-1,-1,-1)$. The linear span of the γ-matrices (generators) contains 1-vectors $\mathbf{x} = \sum x^i\gamma_i$. Define $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ and note that $\gamma_5^2 = -\mathbf{1}$. If we define *new generators* $\alpha_i := \gamma_i\gamma_5$ which are 3-vectors(!), it is easily checked that they nevertheless fulfill $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\eta_{ij}\mathbf{1}$. They might be called *vectors* on an equal right.

Define the map $\underline{\gamma_5} : C\ell_{1,3} \mapsto C\ell_{1,3}$, $\mathbf{x} \mapsto \mathbf{x}' := \mathbf{x}\gamma_5$, lifted to $C\ell_{1,3}$. We have thus defined two *different* Clifford maps $\gamma : V \mapsto C\ell_{1,3}$ and $\gamma' : V \mapsto C\ell_{1,3}$ with $\gamma' := \underline{\gamma_5} \circ \gamma$. That is one can't know for sure which elements are 'vectors' even in this case.

We emphasized earlier that we did not expect the interpretation and the mathematical aspects of classical Clifford algebras to change in such a transformation. However, see [16] for a far more elaborate application of a similar situation where *both* gradings are used.

5.2 Example 2

In this example we examine the split case $C\ell_{2,2} \simeq C\ell_{1,1} \otimes C\ell_{1,1}$ and show the existence and irreducibility of an 8-dimensional representation not known in the classical representation theory of Clifford algebras.

We start with $C\ell_{1,1}(B)$ where B is given as

$$B := \begin{pmatrix} 1 & a \\ 0 & -1 \end{pmatrix}. \tag{44}$$

If a is zero, we have two choices for an idempotent element generating a spinor space:

$$\mathbf{f}_{11}^{-} := \frac{1}{2}(\mathbf{1} + \mathbf{e}_1), \qquad \mathbf{f}_{11}^{+} := \frac{1}{2}(\mathbf{1} + \mathbf{e}_1 \wedge \mathbf{e}_2)\,. \tag{45}$$

A spinor basis can be found in both cases by left multiplying by $\mathbf{e}_2$ which yields $\mathcal{S}^{\pm} =< \mathbf{f}_{11}^{\pm}, \mathbf{e}_2\mathbf{f}_{11}^{\pm} >$. The spinor spaces $\mathcal{S}^{\pm}$ are 2-dimensional and the Clifford elements are represented as 2×2 matrices. If a is not zero, an analogous construction runs through.

Now let us put together two such algebras, as shown in [52], generated by $C\ell_{1,1} =< \mathbf{e}_1, \mathbf{e}_2 >$ and $C\ell_{1,1} =< \mathbf{e}_3, \mathbf{e}_4 >$. The bilinear form B which reduces in both cases to the above setting *and* which contains connecting elements is

$$B := \begin{pmatrix} 1 & a & n_{11} & n_{12} \\ 0 & -1 & n_{21} & n_{22} \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{46}$$

We expect the n_{ij} parameters to govern the deformation of the tensor product in the decomposition theorem.

Searching with CLIFFORD for idempotents in this general case yields the following fact. Let λ be a fixed parameter. Among six choices for an idempotent $\mathbf{f}$, we found

$$\mathbf{f} := \frac{1}{2}(\mathbf{1} + X_1) = \frac{1}{4}(2 + \lambda\, a)\mathbf{1} + \frac{1}{4}\sqrt{4 - \lambda^2 a^2 - 4\lambda^2}\,\mathbf{e}_1 + \frac{1}{2}\lambda\,\mathbf{e}_1 \wedge \mathbf{e}_2$$

where X_1 is one of six different, non-trivial, and general elements X in $C\ell(B,V)$ satisfying $X^2 = \mathbf{1}$. This is an *indecomposable idempotent* which therefore generates an *irreducible* 8 *dimensional representation* since the regular representation of $C\ell(B,V)$ is of dimension 16. This fact depends on the appearance of the non-zero n_{ij} parameters. It was proved by brute force that none of the remaining five non-trivial elements $X_i, i = 2, \ldots, 6$, and squaring to $\mathbf{1}$ commuted with X_1. Thus, the search showed that there is no second Clifford element $X_2 \neq X_1$ which would square to $\mathbf{1}$ and which would commute with X_1. Such an element would be necessary to decompose $\mathbf{f}$ into a product $\mathbf{f} = \prod_i \frac{1}{2}(\mathbf{1} + X_i)$ where $X_i X_j = X_j X_i$ and $X_i^2 = \mathbf{1}$. Since this type of reasoning can be used to classify Clifford algebras [19] we have found a way to classify quantum Clifford algebras.

This type of an indecomposable exotic representation will occur in the next example of a physical model and is thereby not academic.

5.3 Example 3

5.3.1 Index doubling

For a simple treatment with a computer algebra, using CLIFFORD package, and for physical reasons not discussed here, see [25, 27, 31], we introduce an index doubling which provides us with a possibility to map the contraction and the wedge onto a new Clifford product in the larger algebra. The benefits of such a treatment are: the associativity of the mapped products, only one algebra product needed during calculations, etc.

Define the self-dual (reflexive) space $\mathbf{V} = V \oplus V^*$ and introduce generators $\mathbf{e}_i$ which span V and V^*

$$V =< \mathbf{e}_1, \ldots, \mathbf{e}_n >, \qquad V^* =< \mathbf{e}_{n+1}, \ldots, \mathbf{e}_{2n} > . \tag{47}$$

In this transition we require that the elements $\mathbf{e}_i$ from V generate a Grassmann sub-algebra and the $\mathbf{e}_{n+1},\ldots,\mathbf{e}_{2n} \in V^*$ are duals which act via the contraction on V. This gives the following conditions on the form $\mathbf{B}:\mathbf{V}\times\mathbf{V}\mapsto\mathbf{R}$:

$$
\begin{array}{ll}
i) & \mathbf{e}_i^2 = \mathbf{e}_i \wedge \mathbf{e}_i \wedge \cdot = 0 \\
ii) & \mathbf{e}_{n+i}^2 = \mathbf{e}_{n+1} \,\lrcorner_{\mathrm{B}}\, \mathbf{e}_{n+i} \,\lrcorner_{\mathrm{B}}\, \cdot = (\mathbf{e}_{n+i} \wedge \mathbf{e}_{n+i}) \,\lrcorner_{\mathrm{B}}\, \cdot = 0 \,.
\end{array} \tag{48}
$$

Thus, with respect to the basis $<\mathbf{e}_1,\ldots,\mathbf{e}_n,\mathbf{e}_{n+1},\ldots,\mathbf{e}_{2n}>$, $\mathbf{B}$ has the following matrix:

$$
\mathbf{B} := \left(\begin{array}{cc} 0 & g \\ g^T & 0 \end{array}\right) + A \;=\; g + A, \tag{49}
$$

where, with an abuse of notation, the symmetric part of $\mathbf{B}$ is again denoted by g. Note that we have introduced here a further freedom since A may be non-trivial also in the V-V and V^*-V^* sectors. This fact has certain physical consequences which were discussed in [27]. The $\mathbf{e}_i$'s from V can be identified with Schwinger sources of quantum field theory [31, 25].

5.3.2 The $U(2)$-model

We simply report here the result from [27] and strongly encourage the reader to consult this work since we quote here only a part of that work which shows the indecomposability of quantum Clifford algebra representations and the therefrom following physical consequences.

Define $C\ell(B,V) \simeq C\ell_{2,2}(B)$ by specifying $\mathbf{V} =< \mathbf{e}_i > = < \mathbf{a}_1^\dagger, \mathbf{a}_2^\dagger, \mathbf{a}_3, \mathbf{a}_4 >$ and

$$
\mathbf{B} := \frac{1}{2}\left(\begin{array}{cc} 0 & 1\!\!1 \\ 1\!\!1 & 0 \end{array}\right) + A, \tag{50}
$$

where $1\!\!1$ is the 2×2 unit matrix and A is an arbitrary but fixed 4×4 antisymmetric matrix with respect to the $\mathbf{e}_i$ or $\mathbf{a}_i$ basis. Note furthermore that the $\mathbf{a}_i$ and $\mathbf{a}_i^\dagger$ fulfill the canonical anti-commutation relations, CAR, of a quantum system: $\{\mathbf{a}_i,\mathbf{a}_j^\dagger\}_+ = \delta_{ij}$. Define furthermore Clifford elements $N, S_i \in \mathsf{R}\oplus\mathbf{V}\wedge\mathbf{V}$, $i\in\{1,2,3\}$ such that the following relations hold:

$$
\begin{array}{c}
[N,a_i]_- \;=\; -a_i, \quad [N,a_i^\dagger]_- \;=\; +a_i^\dagger, \quad N^\dagger \;=\; N, \\
[S_k,a_i]_- \;=\; \sigma_{ij}a_j, \quad \text{h.c.}, \quad k\in\{1,2,3\}, \\
[S_k,N]_- \;=0, \quad [S_k,S_l]_- \;=\; i\epsilon_{klm}\,S_m, \quad S_K^\dagger = S_k \,,
\end{array} \tag{51}
$$

where $\dagger$ is the anti-involutive map (includes a product reversion) interchanging $\mathbf{a}_i \leftrightarrow \mathbf{a}_i^\dagger$. This is the $U(2)$ algebra if $A\equiv 0$.

Define a 'vacuum', for a discussion see [27], simply be defining the expectation function –linear functional– as the projector onto the scalar part $<\,.\,>_0^A$ which depends now explicitly on A. In a physicist's notation $<0\mid\hat{\mathcal{H}}\mid 0>\simeq\,<\mathcal{H}>_0^A$ for any operator $\hat{\mathcal{H}}$ resp. Clifford element $\mathcal{H}$.

An algebraic analysis which coincides in the positive definite case with C^*-algebraic results shows that this linear functional called 'vacuum' can be uniquely decomposed in certain extremal, that is indecomposable, states. Denoting these states as spinor like $\mathcal{S}_1, \mathcal{S}_2$ and exotic $\mathcal{E}$ we obtain the following identity:

$$< . >_0^A = \lambda_1 < . >^{\mathcal{S}_1} + \lambda_2 < . >^{\mathcal{S}_2} + \lambda_3 < . >^{\mathcal{E}}, \quad \sum \lambda_i = 1. \tag{52}$$

Since the regular representation of $C\ell_{2,2}(B)$ is 16 dimensional and we find dim $\mathcal{S}_1 =$ dim $\mathcal{S}_2 = 4$, dim $\mathcal{E} = 8$ this is a direct sum decomposition into irreducible representations. The 'classical' case would have led to four representations of the spinor type each 4 dimensional. The indecomposable exotic representation obtained from $< . >^{\mathcal{E}}$ is therefore new and it is a direct outcome of the structure of the quantum Clifford algebra, see previous example. This representation decomposes into two spinor like parts if A vanishes identically $A \equiv 0$.

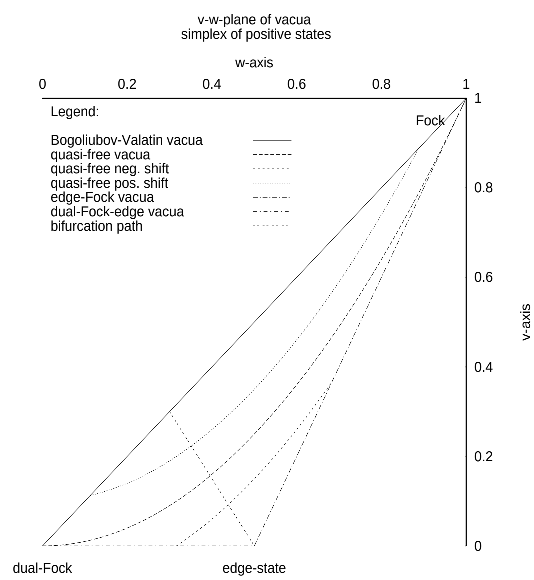

Figure 1: $C\ell(B)$-deformation of $U(2)$ algebra

In [27] we obtained a v-w-plane of vacua while implementing the $\sum \lambda_i = 1$ condition and renaming of variables into v, w. There it was shown, see Figure 1, that we find free systems of Fock and dual-Fock type which constitutes the spinor representations $\mathcal{S}_1$, $\mathcal{S}_2$ and that the line connecting them contains

Bogoliubov-transformed ground-states of BCS-superconductivity. Quasi free, that is correlation free, states are on the displayed parabola. In the exotic state one finds spin 1 and spin 0 components which are beyond Bogoliubov transformations. Every choice of A fixes *exactly* one particular state in the v-w-plane. Hence, we have solved the problem of finding an algebraic condition on which side of the Clebsch-Gordan identity $\frac{1}{2} \otimes \frac{1}{2} = \mathbf{0} \oplus \mathbf{1}$ our algebraic system *has to be* treated.

Our model, even if only marginally discussed, shows all features we want to see in the composite and multi-particle theory. Moreover, exotic representations which describe 'bound objects' not capable of a decomposition are *beyond* the treatment in [20] which mimics in Clifford algebraic terms the usual tensor method which generically bears this problem. In this context we refer to the interesting work of Daviau [18] on de Broglie's spin fusion theory [9] and to the joint works with Stumpf and Dehnen [23, 26] which are connected with algebraic composite theories.

Acknowledgment

The first author (BF) acknowledges a travel grant of the DFG and a critical reading of the manuscript by Th. Konrad.

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