A Classification of Clifford Algebras as Images of Group Algebras of Salingaros Vee Groups

by

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Abstract

- Salingaros (1981, 1982, 1984) defined five families N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k} and S_k of finite 2-groups related to Clifford algebras $C\ell_{p,q}$. For each $k \geq 1$, the group N_{2k-1} is a central product $(D_8)^{\circ k}$ of k copies of the dihedral group D_8 and the group N_{2k} is a central product $(D_8)^{\circ(k-1)} \circ Q_8$, where Q_8 is the quaternion group. Both groups N_{2k-1} and N_{2k} are extra-special.
- $\Omega_{2k-1} \cong N_{2k-1} \circ (C_2 \times C_2)$, $\Omega_{2k} \cong N_{2k} \circ (C_2 \times C_2)$, and $S_k \cong N_{2k-1} \circ C_4 \cong N_{2k} \circ C_4$, where C_2 and C_4 are cyclic groups of order 2 and 4, respectively (cf. Brown (2015)).
- Chernov (2001) observed that a Clifford algebra $C\ell_{p,q}$ could be obtained as a homomorphic image of a group algebra $\mathbb{R}[G]$ should there exist a suitable finite 2-group with generators fulfilling certain relations. As an example, he showed that $C\ell_{0,2} \cong \mathbb{R}[Q_8]/\mathcal{J}$ while $C\ell_{1,1} \cong \mathbb{R}[D_8]/\mathcal{J}$, where in each case $\mathcal{J}$ is an ideal in the respective group algebra generated by $1+\tau$ for some central group element τ of order 2.

Abstract - continued

- Walley (2017) showed that also $C\ell_{2,0} \cong \mathbb{R}[D_8]/\mathcal{J}$ and that eight-dimensional Clifford algebras can be represented as follows:

$$C\ell_{0,3} \cong \mathbb{R}[\Omega_2]/\mathcal{J}, \quad C\ell_{2,1} \cong \mathbb{R}[\Omega_1]/\mathcal{J}, \quad C\ell_{1,2} \cong C\ell_{3,0} \cong \mathbb{R}[S_1]/\mathcal{J}.$$

- In each case one needs to carefully define a surjective map from the group algebra to the Clifford algebra with kernel equal to the ideal $(1+\tau)$.
- One observes that for each $n = p + q \geq 0$ the number of non-isomorphic Salingaros groups of order 2^{n+1} equals the number of isomorphism classes of Clifford algebras $C\ell_{p,q}$ of dimension 2^n (see Periodicity of Eight table in Lounesto (2001)).
- The objective of this work is to prove the following theorem:

Main Theorem

Every Clifford algebra $C\ell_{p,q}$ is isomorphic to a quotient of a group algebra $\mathbb{R}[G]$, where G is one of Salingaros groups N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k} or S_k of order 2^{p+q+1}, modulo an ideal $\mathcal{J} = (1+\tau)$ generated by $1+\tau$ for some central element of order 2.

Abstract - continued

- For example, Salingaros groups N_3 and N_4 are sufficient to give two isomorphism classes of sixteen-dimensional Clifford algebras, namely:

$$C\ell_{0,4} \cong C\ell_{1,3} \cong C\ell_{4,0} \cong \mathbb{R}[N_4]/\mathcal{J}, \quad C\ell_{2,2} \cong C\ell_{3,1} \cong \mathbb{R}[N_3]/\mathcal{J}.$$

- This approach to the Periodicity of Eight of Clifford algebras should allow to apply the representation theory and characters of finite groups to Clifford algebras.
- For example, as a consequence of the well-known fact that up to an isomorphism there are exactly two non-isomorphic non-Abelian groups of order eight provides a group-theoretic explanation why there are exactly two isomorphism classes of Clifford algebras of dimension four.

Keywords: 2-group, central product, Clifford algebra, extra-special group, group algebra, Salingaros vee group

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Definition of a group algebra $\mathbb{F}[G]$

Definition 1

Let G be a finite group and let $\mathbb{F}$ be a field (usually $\mathbb{R}$ or $\mathbb{C}$). Then the **group algebra** $\mathbb{F}[G]$ is the vector space

$$\mathbb{F}[G] = \left\{ \sum_{g \in G} \lambda_g g, \ \lambda_g \in \mathbb{F} \right\} \tag{1}$$

with multiplication defined as

$$\left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g,h \in G} \lambda_g \mu_h (gh) = \sum_{g \in G} \sum_{h \in G} (\lambda_h \mu_{h^{-1}g}) g \tag{2}$$

where all $\lambda_g, \mu_h \in \mathbb{F}$. (James and Liebeck [11], Passman [16])

Definition 2

Let p be a prime. A group G is a p-**group** if every element in G is of order p^k for some $k \geq 1$. So, any finite group G of order p^n is a p-group.

Two important groups Q_8 and D_8 of order 8

- The quaternionic group Q_8:

$$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$$
$$= \langle I, J, \tau \mid \tau^2 = 1, I^2 = J^2 = \tau, IJ = \tau JI \rangle \quad (I = a, J = b, \tau = a^2)$$

so $|a^2| = 2$, $|a| = |a^3| = |b| = |ab| = |a^2b| = |a^3b| = 4$, hence its order structure is $[1, 1, 6]$, and $Z(Q_8) = \{1, a^2\} \cong C_2$. Here, $\tau = a^2 \in Z(Q_8)$.

- The dihedral group D_8 (the symmetry group of a square):

$$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$$
$$= \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^{-1} \rangle \quad (\tau = a, \sigma = b)$$

so $|a^2| = |b| = |ab| = |a^2b| = |a^3b| = 2$, $|a| = |a^3| = 4$, hence its order structure is $[1, 5, 2]$, and $Z(D_8) = \{1, a^2\} \cong C_2$. Here, $\tau = b$, $\sigma = a$, hence, $\sigma^2 = a^2 \in Z(D_8)$.

Constructing $\mathbb{H} = C\ell_{0,2}$ as $\mathbb{R}[Q_8]/\mathcal{J}$

Example 1

Define an algebra map ψ *from the group algebra* $\mathbb{R}[Q_8] \to \mathbb{H} = \{1, \mathbf{i}, \mathbf{j}, \mathbf{ij}\}$:

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad I \mapsto \mathbf{i}, \quad J \mapsto \mathbf{j}, \tag{3}$$

Then, $\mathcal{J} = \ker \psi = (1+\tau)$ *for a central involution* $\tau = a^2$ *in* Q_8, *so* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[Q_8] \to \mathbb{R}[Q_8]/\mathcal{J}$ *be the natural map* $u \mapsto u + \mathcal{J}$. *There exists an isomorphism* $\varphi : \mathbb{R}[Q_8]/\mathcal{J} \to \mathbb{H}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\pi(I^2) = I^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(I^2)) = \psi(\tau) = -1 = (\psi(I))^2 = \mathbf{i}^2,$$
$$\pi(J^2) = J^2 + \mathcal{J} = \tau + \mathcal{J} \text{ and } \varphi(\pi(J^2)) = \psi(\tau) = -1 = (\psi(J))^2 = \mathbf{j}^2,$$
$$\pi(IJ + JI) = IJ + JI + \mathcal{J} = (1+\tau)JI + \mathcal{J} = \mathcal{J} \text{ and}$$
$$\varphi(\pi(IJ + JI)) = \psi(0) = 0 = \psi(I)\psi(J) + \psi(J)\psi(I) = \mathbf{ij} + \mathbf{ji}.$$

Thus, $\mathbb{R}[Q_8]/\mathcal{J} \cong \psi(\mathbb{R}[Q_8]) = \mathbb{H} \cong C\ell_{0,2}$ *provided the central involution* τ *is mapped into* -1. *(see also Chernov [5])*

Constructing $C\ell_{1,1}$ as $\mathbb{R}[D_8]/\mathcal{J}$

Example 2

Define an algebra map ψ *from the group algebra* $\mathbb{R}[D_8] \to C\ell_{1,1}$ *such that:*

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_2, \tag{4}$$

where $C\ell_{1,1} = \operatorname{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$. *Then,* $\ker \psi = (1 + \sigma^2)$ *where* σ^2 *is a central involution* a^2 *in* D_8. *Let* $\mathcal{J} = (1 + \sigma^2)$. *Thus,* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ *be the natural map* $u \mapsto u + \mathcal{J}$. *There exists an isomorphism* $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{1,1}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\begin{gathered}
\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \text{ and } \varphi(\pi(\tau^2)) = \psi(1) = \psi(\tau^2) = (\mathbf{e}_1)^2 = 1, \\
\pi(\sigma^2) = \sigma^2 + \mathcal{J} \text{ and } \varphi(\pi(\sigma^2)) = \psi(\sigma^2) = \psi(-1) = (\mathbf{e}_2)^2 = -1, \\
\pi(\tau\sigma + \sigma\tau) = \tau\sigma + \sigma\tau + \mathcal{J} = \sigma\tau(1 + \sigma^2) + \mathcal{J} = \mathcal{J} \text{ and} \\
\varphi(\pi(\tau\sigma + \sigma\tau) = \psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \psi(0) = \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.
\end{gathered}$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ *provided the central involution* σ^2 *is mapped into* -1.

Constructing $C\ell_{2,0}$ as $\mathbb{R}[D_8]/\mathcal{J}$

Example 3

Define an algebra map ψ *from the group algebra* $\mathbb{R}[D_8] \to C\ell_{2,0}$ *such that:*

$$1 \mapsto 1, \quad \tau \mapsto \mathbf{e}_1, \quad \sigma \mapsto \mathbf{e}_1\mathbf{e}_2, \tag{5}$$

where $C\ell_{2,0} = \operatorname{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$. *Then,* $\ker\psi = (1+\sigma^2)$ *where* σ^2 *is a central involution* a^2 *in* D_8. *Let* $\mathcal{J} = (1+\sigma^2)$. *Thus,* $\dim_{\mathbb{R}} \mathcal{J} = 4$ *and* ψ *is surjective. Let* $\pi : \mathbb{R}[D_8] \to \mathbb{R}[D_8]/\mathcal{J}$ *be the natural map* $u \mapsto u + \mathcal{J}$. *There exists an isomorphism* $\varphi : \mathbb{R}[D_8]/\mathcal{J} \to C\ell_{2,0}$ *such that* $\varphi \circ \pi = \psi$ *and*

$$\pi(\tau^2) = \tau^2 + \mathcal{J} = 1 + \mathcal{J} \textit{ and } \varphi(\pi(\tau^2)) = \psi(\tau^2) = \psi(1) = (\mathbf{e}_1)^2 = 1,$$
$$\pi(\sigma^2) = \sigma^2 + \mathcal{J} \textit{ so } \varphi(\pi(\sigma^2)) = \psi(-1) = (\mathbf{e}_1\mathbf{e}_2)^2 = -1, \textit{ so } (\mathbf{e}_2)^2 = 1 \textit{ since}$$
$$\varphi(\pi(\tau\sigma + \sigma\tau)) = \varphi(\tau\sigma + \sigma\tau + \mathcal{J}) = \varphi(\sigma\tau(1+\sigma^2) + \mathcal{J}) = \varphi(\mathcal{J}) \textit{ and}$$
$$\psi(\tau)\psi(\sigma) + \psi(\sigma)\psi(\tau) = \psi(0) = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = 0, \textit{ so } \mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = 0.$$

Thus, $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{2,0}$ *provided the central involution* σ^2 *is mapped into* -1.

Summary of projective constructions of $C\ell_{0,2}$ and $C\ell_{1,1}$

Notice first that each group $N_2 = Q_8$ and $N_1 = D_8$ can be written as follows:

- The quaternionic group Q_8:

$$Q_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0,1\},\ k = 0,1,2\}$$

where $\tau = a^2$ is the central involution in Q_8, $g_1 = a$, and $g_2 = b$. Thus,

$$(g_1)^2 = a^2 = \tau, \quad (g_2)^2 = b^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

Observe that $|g_1| = |g_2| = 4$ and $\mathbb{R}[Q_8]/\mathcal{J} \cong C\ell_{0,2}$ where $\mathcal{J} = (1+\tau)$.

- The dihedral group D_8:

$$D_8 = \{\tau^{\alpha_0} g_1^{\alpha_1} g_2^{\alpha_2} \mid \alpha_k \in \{0,1\},\ k = 0,1,2\}$$

where $\tau = a^2$ is the central involution in D_8, $g_1 = b$, and $g_2 = a$. Thus,

$$(g_1)^2 = b^2 = 1, \quad (g_2)^2 = a^2 = \tau, \quad \tau g_1 g_2 = g_2 g_1.$$

Observe that $|g_1| = 2$, $|g_2| = 4$ and $\mathbb{R}[D_8]/\mathcal{J} \cong C\ell_{1,1}$ where $\mathcal{J} = (1+\tau)$.

Reformulated Chernov's Theorem [5]

Theorem 3

Let G be a finite 2-group of order 2^{1+n} generated by a central involution τ and additional elements $g_1, \ldots, g_n$, which satisfy the following relations:

$$\tau^2 = 1, \quad (g_1)^2 = \cdots = (g_p)^2 = 1, \quad (g_{p+1})^2 = \cdots = (g_{p+q})^2 = \tau, \tag{6}$$

$$\tau g_j = g_j \tau, \quad g_i g_j = \tau g_j g_i, \quad i,j = 1, \ldots, n = p+q, \tag{7}$$

Let $\mathcal{J} = (1+\tau)$ be an ideal in the group algebra $\mathbb{R}[G]$ and let $C\ell_{p,q}$ be the universal real Clifford algebra generated by $\{\mathbf{e}_k\}$, $k = 1, \ldots, n = p+q$, where

$$\mathbf{e}_i^2 = Q(\mathbf{e}_i) \cdot 1 = \varepsilon_i \cdot 1 = \begin{cases} 1 & \text{for } 1 \leq i \leq p; \\ -1 & \text{for } p+1 \leq i \leq p+q; \end{cases} \tag{8a}$$

$$\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = 0, \quad i \neq j, \quad 1 \leq i,j \leq n. \tag{8b}$$

Then, (a) $\dim_{\mathbb{R}} \mathcal{J} = 2^n$; *(b) There exists a surjective algebra homomorphism ψ from the group algebra $\mathbb{R}[G]$ to $C\ell_{p,q}$ so that* $\ker \psi = \mathcal{J}$ *and $\mathbb{R}[G]/\mathcal{J} \cong C\ell_{p,q}$.*

Proof of Theorem 3

Proof.

Observe that $G = \{\tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n}\} \mid \alpha_k \in \{0,1\},\ k = 0, 1, \ldots, n\}$. The existence of a central involution τ is guaranteed by a well-known fact that the center of any p-group is nontrivial, and by Cauchy Theorem. (Rotman [17]) Define an algebra homomorphism $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ such that

$$1 \mapsto 1, \quad \tau \mapsto -1, \quad g_j \mapsto \mathbf{e}_j, \quad j = 1, \ldots, n. \tag{9}$$

Clearly, $\mathcal{J} \subset \ker \psi$. Let $u \in \mathbb{R}[G]$. Then, $u = \sum_\alpha \lambda_\alpha \tau^{\alpha_0} g_1^{\alpha_1} \cdots g_n^{\alpha_n} = u_1 + \tau u_2$ where $u_i = \sum_{\widetilde{\alpha}} \lambda_{\widetilde{\alpha}}^{(i)} g_1^{\alpha_1} \cdots g_n^{\alpha_n}$, $i = 1, 2$, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ and $\widetilde{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Thus, if $u \in \ker \psi$, then

$$\psi(u) = \sum_{\widetilde{\alpha}} (\lambda_{\widetilde{\alpha}}^{(1)} - \lambda_{\widetilde{\alpha}}^{(2)}) \mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} = 0$$

implies $\lambda_{\widetilde{\alpha}}^{(1)} = \lambda_{\widetilde{\alpha}}^{(2)}$ since $\{\mathbf{e}_1^{\alpha_1} \cdots \mathbf{e}_n^{\alpha_n} \mid (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_2)^n\}$ is a basis in $C\ell_{p,q}$.

Proof of Theorem 3 (Cont.)

Hence,

$$u = (1+\tau)\sum_{\widetilde{\alpha}} \lambda^{(1)}_{\widetilde{\alpha}} g_1^{\alpha_1}\cdots g_n^{\alpha_n} \in \mathcal{J}.$$

Thus, $\dim_{\mathbb{R}} \ker\psi = 2^n$, $\ker\psi = \mathcal{J}$, $\dim_{\mathbb{R}} \mathbb{R}[G]/\mathcal{J} = 2^{1+n} - 2^n = 2^n$, so ψ is surjective. Let $\varphi : \mathbb{R}[G]/\mathcal{J} \to C\ell_{p,q}$ be such that $\varphi \circ \pi = \psi$ where $\pi : \mathbb{R}[G] \to \mathbb{R}[G]/\mathcal{J}$ is the natural map. Then, since $\psi(g_j) = \mathbf{e}_j$, $\pi(g_j) = g_j + \mathcal{J}$, we have $\varphi(\pi(g_j)) = \varphi(g_j + \mathcal{J}) = \psi(g_j) = \mathbf{e}_j$ and

$$\begin{aligned}\pi(g_j)\pi(g_i) + \pi(g_i)\pi(g_j) &= (g_j+\mathcal{J})(g_i+\mathcal{J}) + (g_j+\mathcal{J})(g_i+\mathcal{J})\\ &= (g_jg_i + g_ig_j) + \mathcal{J} = (1+\tau)g_jg_i + \mathcal{J} = \mathcal{J}\end{aligned}$$

for $i \neq j$ since $g_ig_j = \tau g_jg_i$ in $\mathbb{R}[G]$, τ is central, and $\mathcal{J} = (1+\tau)$. Thus, $g_j + \mathcal{J}, g_i + \mathcal{J}$ anticommute in $\mathbb{R}[G]/\mathcal{J}$ when $i \neq j$. Also, when $i = j$,

$$\pi(g_i)\pi(g_i) = (g_i+\mathcal{J})(g_i+\mathcal{J}) = (g_i)^2 + \mathcal{J} = \begin{cases} 1+\mathcal{J}, & 1 \le i \le p;\\ \tau+\mathcal{J}, & p+1 \le i \le n;\end{cases}$$

Proof of Theorem 3 (Cont.)

Observe, that

$$\tau + \mathcal{J} = (-1) + (1+\tau) + \mathcal{J} = (-1) + \mathcal{J} \text{ in } \mathbb{R}[G]/\mathcal{J}.$$

To summarize, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is generated by the cosets $g_i + \mathcal{J}$ which satisfy these relations:

$$(g_j + \mathcal{J})(g_i + \mathcal{J}) + (g_j + \mathcal{J})(g_i + \mathcal{J}) = \mathcal{J},$$

$$(g_i)^2 + \mathcal{J} = \begin{cases} 1 + \mathcal{J}, & 1 \le i \le p; \\ (-1) + \mathcal{J}, & p+1 \le i \le n; \end{cases}$$

Thus, the factor algebra $\mathbb{R}[G]/\mathcal{J}$ is a Clifford algebra isomorphic to $C\ell_{p,q}$ provided $\mathcal{J} = (1+\tau)$ for the central involution τ in G. □

Note: Example 3 shows that the map $\psi : \mathbb{R}[G] \to C\ell_{p,q}$ need not be defined as in (9). This allows to define different surjective ψ maps from the same group algebra $\mathbb{R}[G]$ to different but isomorphic Clifford algebras, e.g., $C\ell_{1,1} \cong C\ell_{2,0}$.

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Salingaros vee groups $G_{p,q} \subset C\ell_{p,q}$

Let $G_{p,q}$ be a finite group in any Clifford algebra $C\ell_{p,q}$ (simple or semisimple) with a binary operation being just the Clifford product, namely:

$$G_{p,q} = \{\pm \mathbf{e}_{\underline{i}} \mid \mathbf{e}_{\underline{i}} \in \mathcal{B} \quad \text{with Clifford product}\}. \tag{10}$$

So, $G_{p,q}$ may be presented as:

$$G_{p,q} = \langle -1, \mathbf{e}_1, \ldots, \mathbf{e}_n \mid \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \ \text{ for } \ i \neq j \ \text{ and } \mathbf{e}_i^2 = \pm 1 \rangle, \tag{11}$$

where $\mathbf{e}_i^2 = 1$ for $1 \leq i \leq p$ and $\mathbf{e}_i^2 = -1$ for $p+1 \leq i \leq n = p+q$. In the following, the elements $\mathbf{e}_{\underline{i}} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$ will be denoted for short as $\mathbf{e}_{i_1 i_2 \cdots i_k}$ for $k \geq 1$ while $\mathbf{e}_{\emptyset}$ will be denoted as 1, the identity element of $G_{p,q}$ (and $C\ell_{p,q}$).
This 2-group of order $2 \cdot 2^{p+q} = 2^{n+1}$ is known as *Salingaros vee group* and has been discussed, for example, by Salingaros [18, 19, 20], Varlamov [22], Helmstetter [9], Abłamowicz and Fauser [2, 3], Maduranga and Abłamowicz [14], and most recently by Brown [4].
$G_{p,q}$ is a discrete subgroup of $\mathbf{Pin}(p,q) \subset \mathbf{\Gamma}_{p,q}$ (Lipschitz group) (Lounesto [12]).

The commutator subgroup G' of a group G

Definition 4

If G is a group and $x, y \in G$, then their **commutator** $[x, y]$ is the element $xyx^{-1}y^{-1}$. If X and Y are subgroups of G, then the **commutator subgroup** $[X, Y]$ of G is defined by

$$[X, Y] = \langle [x, y] \mid x \in X, y \in Y \rangle.$$

In particular, the **derived subgroup** G' of G is defined as $G' = [G, G]$.

Proposition 1

Let G be a group.

(i) *G' is a normal subgroup of G, and G/G' is abelian.*

(ii) *If $H \lhd G$ and G/H is abelian, then $G' \subseteq H$.*

(Rotman [17])

Subgroup $G_{p,q}(f)$ of $G_{p,q}$

- $G_{p,q}$ – Salingaros vee group of order $|G_{p,q}| = 2^{1+p+q}$
- $G'_{p,q} = \{1, -1\}$ – the commutator subgroup of $G_{p,q}$
- Let $\mathcal{O}(f)$ be the orbit of a primitive idempotent f under the conjugate action of $G_{p,q}$, and let $G_{p,q}(f)$ be the stabilizer of f. Let

  $$N = |\mathcal{F}| = [G_{p,q} : G_{p,q}(f)] = |\mathcal{O}(f)| = |G_{p,q}|/|G_{p,q}(f)| = 2 \cdot 2^{p+q}/|G_{p,q}(f)|$$

  then $N = 2^k$ (resp. $N = 2^{k-1}$) for simple (resp. semisimple) $C\ell_{p,q}$ where $k = q - r_{q-p}$ and $[G_{p,q} : G_{p,q}(f)]$ is the index of $G_{p,q}(f)$ in $G_{p,q}$.
- $G_{p,q}(f) \lhd G_{p,q}$ and $|G_{p,q}(f)| = 2^{1+p+r_{q-p}}$ (resp. $|G_{p,q}(f)| = 2^{2+p+r_{q-p}}$) for simple (resp. semisimple) $C\ell_{p,q}$.
- The set of commuting monomials $\mathcal{T} = \{\mathbf{e}_{\underline{i}_1}, \ldots, \mathbf{e}_{\underline{i}_k}\}$ (squaring to 1) in the primitive idempotent $f = \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_1}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\underline{i}_k})$ is point-wise stabilized by $G_{p,q}(f)$.

(for more subgroups of $G_{p,q}$ see [1, 2, 3]).

Summary of some basic properties of $G_{p,q}$

Summary of some basic properties of $G_{p,q}$

- $|G_{p,q}| = 2^{1+p+q}$, $|G'_{p,q}| = 2$ because $G'_{p,q} = \{\pm 1\}$
- $G_{p,q}$ is not simple as it has a normal subgroup of order 2^m for every $m \leq 1+p+q$ (because every p-group of order p^n has a normal subgroup of order p^m for every $m \neq n$).
- The center of any group $G_{p,q}$ is non-trivial since $2 \mid |Z(G_{p,q})|$ and so every group $G_{p,q}$ has a central element τ of order 2. It is well-known that for any prime p and a finite p-group $G \neq \{1\}$, the center of G is non-trivial. (Rotman [17])
- Every element of $G_{p,q}$ is of order 1, 2, or 4.
- Since $[G_{p,q} : G'_{p,q}] = |G_{p,q}|/|G'_{p,q}| = 2^{p+q}$, each $G_{p,q}$ has 2^{p+q} linear characters. (James and Liebeck [11]).
- The number N of conjugacy classes in $G_{p,q}$, hence, the number of irreducible inequivalent representations of $G_{p,q}$, is $1 + 2^{p+q}$ (resp. $2 + 2^{p+q}$) when $p+q$ is even (resp. odd). (Maduranga [13])

Summary of some basic properties of $G_{p,q}$ (Cont.)

- We have the following (see also Varlamov [22]):

Theorem 5

Let $G_{p,q} \subset C\ell_{p,q}^{*}$. Then,

$$Z(G_{p,q}) = \begin{cases} \{\pm 1\} \cong C_2 & \textit{if } p-q \equiv 0,2,4,6 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong C_2 \times C_2 & \textit{if } p-q \equiv 1,5 \pmod 8; \\ \{\pm 1, \pm\beta\} \cong C_4 & \textit{if } p-q \equiv 3,7 \pmod 8. \end{cases} \tag{12}$$

as a consequence of $Z(C\ell_{p,q}) = \{1\}$ (resp. $\{1,\beta\}$) when $p+q$ is even (resp. odd) where $\beta = \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_n$, $n = p+q$, is the unit pseudoscalar in $C\ell_{p,q}$.

Note: In the above, C_n denotes a cyclic group of order n.

Summary of some basic properties of $G_{p,q}$ (Cont.)

- Salingaros' notation: $N_{2k-1}, N_{2k}, \Omega_{2k-1}, \Omega_{2k}, S_k$:

Table 1: Five isomorphism classes of vee groups $G_{p,q}$ in $C\ell^*_{p,q}$

Group G	$Z(G)$	Group order	dim $C\ell_{p,q}$	$Z(C\ell_{p,q})$
N_{2k-1}	C_2	2^{2k+1}	2^{2k}	$\{1\}$
N_{2k}	C_2	2^{2k+1}	2^{2k}	$\{1\}$
Ω_{2k-1}	$C_2 \times C_2$	2^{2k+2}	2^{2k+1}	$\{1, \beta\}$
Ω_{2k}	$C_2 \times C_2$	2^{2k+2}	2^{2k+1}	$\{1, \beta\}$
S_k	C_4	2^{2k+2}	2^{2k+1}	$\{1, \beta\}$

$$\begin{aligned}
N_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q}, \ p-q \equiv 0,2 \pmod 8, \ \mathbb{K} \cong \mathbb{R};\\
N_{2k} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q}, \ p-q \equiv 4,6 \pmod 8, \ \mathbb{K} \cong \mathbb{H};\\
\Omega_{2k-1} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q}, \ p-q \equiv 1 \pmod 8, \ \mathbb{K} \cong \mathbb{R} \oplus \mathbb{R};\\
\Omega_{2k} &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q}, \ p-q \equiv 5 \pmod 8, \ \mathbb{K} \cong \mathbb{H} \oplus \mathbb{H};\\
S_k &\leftrightarrow G_{p,q} \subset C\ell^*_{p,q}, \ p-q \equiv 3,7 \pmod 8, \ \mathbb{K} \cong \mathbb{C}.
\end{aligned}$$

(Brown [4], Lounesto [12], Salingaros [18, 19, 20], Varlamov [22])

Vee groups $G_{p,q}$ of low orders 4, 8, 16

The first few vee groups $G_{p,q}$ corresponding to Clifford algebras $C\ell_{p,q}$ in dimensions $p+q=1,2,3$, are:

$$\begin{aligned} \text{Groups of order 4:} \quad & G_{1,0} = D_4, \quad G_{0,1} = C_4, \\ \text{Groups of order 8:} \quad & G_{2,0} = D_8 = N_1, \quad G_{1,1} = D_8 = N_1, \quad G_{0,2} = Q_8 = N_2, \\ \text{Groups of order 16:} \quad & G_{3,0} = S_1, \quad G_{2,1} = \Omega_1, \quad G_{1,2} = S_1, \quad G_{0,3} = \Omega_2. \end{aligned}$$

where D_8 is the dihedral group of a square, Q_8 is the quaternionic group, and $D_4 \cong C_2 \times C_2$. For a construction of inequivalent irreducible representations and characters of these groups see Maduranga and Abłamowicz [14] and Maduranga [13].

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Definitions: elementary abelian group, extra-special groups

Definition 6 (Gorenstein [8])

A finite abelian p-group is **elementary abelian** if every nontrivial element has order p.

Example 7 ($D_4 = C_2 \times C_2$ is elementary abelian)

$(C_p)^k = C_p \times \cdots \times C_p$ (k-times), in particular, $C_2 \times C_2$ is elementary abelian.

Definition 8 (Dornhoff [6])

A finite p-group P is **extra-special** if

(i) $P' = Z(P)$,

(ii) $|P'| = p$, and

(iii) P/P' is elementary abelian.

D_8 and Q_8 are extra-special and non-isomorphic

Example 9 (D_8 is extra-special)

$D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1} \rangle$ is extra-special because:

- $Z(D_8) = D_8' = [D_8, D_8] = \langle a^2 \rangle$, $|Z(D_8)| = 2$,
- $D_8/D_8' = D_8/Z(D_8) = \langle D_8', aD_8', bD_8', abD_8' \rangle \cong C_2 \times C_2$.
- Order structure: $[1, 5, 2]$
- $D_8 \cong C_4 \rtimes C_2 \cong (C_2 \times C_2) \rtimes C_2$ (semi-direct products)

Example 10 (Q_8 is extra-special)

$Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ is extra-special because:

- $Z(Q_8) = Q_8' = [Q_8, Q_8] = \langle a^2 \rangle$, $|Z(Q_8)| = 2$,
- $Q_8/Q_8' = Q_8/Z(Q_8) = \langle Q_8', aQ_8', bQ_8', abQ_8' \rangle \cong C_2 \times C_2$.
- Order structure: $[1, 1, 6]$
- Q_8 is not a semi-direct product of any of its subgroups (cf. Brown [4])

Definitions: internal and external central products of groups

Definition 11 (Gorenstein [8])

(i) A group G is an **internal central product** of two subgroups H and K if:

(a) $[H,K] = \langle 1 \rangle$;
(b) $G = HK$;

Note: $H, K \lhd G$, $Z(H), Z(K) < Z(G)$ and if G is a 2-group with $|Z(G)| = 2$, then $Z(G) = Z(H) = Z(K) \cong C_2$.

(ii) A group G is an **external central product** $H \circ K$ of two groups H and K with $H_1 \leq Z(H)$ and $K_1 \leq Z(K)$ if there exists an isomorphism $\theta : H_1 \to K_1$ such that G is $(H \times K)/N$ where

$$N = \{(h, \theta(h^{-1})) \mid h \in H_1\}.$$

Note: $N \lhd (H \times K)$ and $|H \circ K| = |H||K|/|N| \leq |H \times K| = |H||K|$.

Extra-special groups as central products

Lemma 12 (Leedham-Green and McKay [15])

An extra-special p-group has order p^{2n+1} for some positive integer n, and is the iterated central product of non-abelian groups of order p^3.

Theorem 13 (Leedham-Green and McKay [15])

There are exactly two isomorphism classes of extra-special groups of order 2^{2n+1} for positive integer n. One isomorphism type arises as the iterated central product of n copies of D_8; the other as the iterated central product of n groups isomorphic to D_8 and Q_8, including at least one copy of Q_8. That is,

1: $D_8 \circ D_8 \circ \cdots \circ D_8$, *or,*

2: $D_8 \circ D_8 \circ \cdots \circ D_8 \circ Q_8$.

where it is understood that these are iterated central products; that is, $D_8 \circ D_8 \circ D_8$ is really $D_8 \circ (D_8 \circ D_8)$ and so on.

Extra-special groups as central products (Cont.)

Lemma 14 (Dornhoff [6])

Let $P_1, \ldots, P_n$ be extra-special p-groups of order p^3. Then there is one and up to isomorphism only one central product of $P_1, \ldots, P_n$ with center of order p. It is extra special of order p^{2n+1} denoted by $P_1 \circ \cdots \circ P_n$, and called the central product of $P_1, \ldots, P_n$.

Lemma 15 (Dornhoff [6])

$Q_8 \circ Q_8$ and $D_8 \circ D_8$ are isomorphic groups of order 32, not isomorphic to $D_8 \circ Q_8$. If C is a cyclic 2-group of order ≥ 4, then $C \circ Q_8 \cong C \circ D_8$.

Note: The above group-theoretic results provide a foundation for Salingaros's Theorem (next).

Salingaros Theorem [20]

Theorem 16

Let $N_1 = D_8$, $N_2 = Q_8$, and $(G)^{\circ k}$ be the iterated central product $G \circ G \circ \cdots \circ G$ (k times) of G. Then, for $k \geq 1$:

1. $N_{2k-1} \cong (N_1)^{\circ k} = (D_8)^{\circ k}$,
2. $N_{2k} \cong (N_1)^{\circ k} \circ N_2 = (D_8)^{\circ(k-1)} \circ Q_8$,
3. $\Omega_{2k-1} \cong N_{2k-1} \circ (C_2 \times C_2) = (D_8)^{\circ k} \circ (C_2 \times C_2)$,
4. $\Omega_{2k} \cong N_{2k} \circ (C_2 \times C_2) = (D_8)^{\circ(k-1)} \circ Q_8 \circ (C_2 \times C_2)$,
5. $S_k \cong N_{2k-1} \circ C_4 \cong N_{2k} \circ C_4 = (D_8)^{\circ k} \circ C_4 \cong (D_8)^{\circ(k-1)} \circ Q_8 \circ C_4$

- C_2, C_4 are cyclic groups of order 2 and 4, respectively;
- D_8 and Q_8 are the dihedral group of a square and the quaternionic group;
- $C_2 \times C_2$ is elementary abelian of order 4;
- N_{2k-1} and N_{2k} are extra-special groups of order 2^{2k+1};
- $\Omega_{2k-1}, \Omega_{2k}, S_k$ are of order 2^{2k+2} (not extra-special).

Vee groups $G_{p,q}$ of orders $16, 32$ as central products

The vee groups $G_{p,q}$ in Clifford algebras $C\ell_{p,q}$ in dimensions $p+q=3,4$:

Order 16:

$$G_{3,0} = G_{1,2} = S_1 = N_1 \circ C_4 = D_8 \circ C_4 = N_2 \circ C_4 = Q_8 \circ C_4,$$
$$G_{2,1} = \Omega_1 = N_1 \circ (C_2 \times C_2) = D_8 \circ (C_2 \times C_2),$$
$$G_{0,3} = \Omega_2 = N_2 \circ (C_2 \times C_2) = Q_8 \circ (C_2 \times C_2),$$

Order 32:

$$G_{4,0} = N_4 = N_1 \circ N_2 = D_8 \circ Q_8 = \langle \mathbf{e}_{123}, \mathbf{e}_4 \rangle \circ \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle,$$
$$G_{3,1} = N_3 = N_1 \circ N_1 = D_8 \circ D_8 = \langle \mathbf{e}_{23}, \mathbf{e}_{24} \rangle \circ \langle \mathbf{e}_{1234}, \mathbf{e}_1 \rangle,$$
$$= N_3 = N_2 \circ N_2 = Q_8 \circ Q_8 = \langle \mathbf{e}_4, \mathbf{e}_{123} \rangle \circ \langle \mathbf{e}_{12}, \mathbf{e}_{13} \rangle,$$
$$G_{2,2} = N_3 = N_1 \circ N_1 = D_8 \circ D_8 = \langle \mathbf{e}_{134}, \mathbf{e}_{24} \rangle \circ \langle \mathbf{e}_3, \mathbf{e}_1 \rangle,$$
$$= N_3 = N_2 \circ N_2 = Q_8 \circ Q_8 = \langle \mathbf{e}_{12}, \mathbf{e}_{134} \rangle \circ \langle \mathbf{e}_3, \mathbf{e}_4 \rangle,$$
$$G_{1,3} = N_4 = N_1 \circ N_2 = D_8 \circ Q_8 = \langle \mathbf{e}_{23}, \mathbf{e}_{12} \rangle \circ \langle \mathbf{e}_{123}, \mathbf{e}_4 \rangle,$$
$$G_{0,4} = N_4 = N_1 \circ N_2 = D_8 \circ Q_8 = \langle \mathbf{e}_{123}, \mathbf{e}_4 \rangle \circ \langle \mathbf{e}_{23}, \mathbf{e}_{12} \rangle.$$

$N_1 = D_8$, $N_2 = Q_8$, N_3, N_4 - extra-special groups,
$C_2 \times C_2$ - elementary abelian group

Table 2: Isomorphism classes of vee groups in Clifford algebras $C\ell_{p,q}$, $n = p + q$.

n \ p-q	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0									N_0								
1								S_0		Ω_0							
2							N_2		N_1		N_1						
3						Ω_2		S_1		Ω_1		S_1					
4					N_4		N_4		N_3		N_3		N_4				
5				S_2		Ω_4		S_2		Ω_3		S_2		Ω_4			
6			N_5		N_6		N_6		N_5		N_5		N_6		N_6		
7		Ω_5		S_3		Ω_6		S_3		Ω_5		S_3		Ω_6		S_3	
8	N_7		N_7		N_8		N_8		N_7		N_7		N_8		N_8		N_7

(Salingaros [18, 19, 20])

Note: There is another way to look at the above table by the Main Theorem (next).

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Theorem 17

Each group $G_{p,q}$ satisfies all hypotheses of Chernov's theorem.

Proof.

Easy observation from the definition of $G_{p,q}$. □

Main Theorem

Every Clifford algebra $C\ell_{p,q}$ is isomorphic to a quotient of a group algebra $\mathbb{R}[G]$, where G is one of Salingaros groups N_{2k-1}, N_{2k}, Ω_{2k-1}, Ω_{2k} or S_k of order 2^{p+q+1}, modulo an ideal $(1+\tau)$ generated by $1+\tau$ for some central element of order 2.

Proof.

Apply Theorem 17 and Chernov's Theorem. □

$\mathbb{Z}_2$-gradation of $\mathbb{R}[G_{p,q}]$

Proposition 2 (Rotman [17])

If G is a p-group of order p^n, then G has a normal subgroup of order p^k for every $k \leq n$.

Corollary 18

Let G be a Salingaros vee group $G_{p,q}$. Then,

(i) *G has a normal subgroup H of index 2.*

(ii) *$G = H \,\dot{\cup}\, Hb$ for some element $b \notin H$ such that $b^2 \in H$.*

(iii) *The group algebra $\mathbb{R}[G]$ is $\mathbb{Z}_2$-graded.*

$\mathbb{Z}_2$-gradation of $\mathbb{R}[G_{p,q}]$ (Cont.)

Proof of (iii):

Since $G = H \dot{\cup} Hb$, we have

$$\mathbb{R}[G] = \left\{ \sum_{h \in H} x_h h + \sum_{h \in H} y_h hb \mid x_h, y_h \in \mathbb{R} \right\}.$$

Let $\mathbb{R}[G]^{(0)} = \{\sum_{h \in H} x_h h \mid x_h \in \mathbb{R}\}$ and $\mathbb{R}[G]^{(1)} = \{\sum_{h \in H} y_h hb \mid y_h \in \mathbb{R}\}$. Then, since $H \lhd G$, $b \notin H$, and $b^2 \in H$, we have

$$\mathbb{R}[G] = \mathbb{R}[G]^{(0)} \oplus \mathbb{R}[G]^{(1)},$$
$$\mathbb{R}[G]^{(i)} \, \mathbb{R}[G]^{(j)} \subseteq \mathbb{R}[G]^{(i+j) \bmod 2}, \quad i, j = 0, 1.$$

□

$\mathbb{Z}_2$-gradation of $\mathbb{R}[D_8]$ and $\mathbb{R}[Q_8]$

Example 19

Let $D_8 = \langle a, b \mid a^4 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, $H = \langle a \mid a^4 = 1\rangle \lhd G$, and

$$\mathbb{R}[D_8]^{(0)} = \left\{x_0 1 + x_1 a + x_2 a^2 + x_3 a^3 \mid x_i \in \mathbb{R}\right\},$$
$$\mathbb{R}[D_8]^{(1)} = \left\{y_0 b + y_1 ab + y_2 a^2 b + y_3 a^3 b \mid y_i \in \mathbb{R}\right\}.$$

Then, $\mathbb{R}[D_8]^{(i)}\mathbb{R}[D_8]^{(j)} \subseteq \mathbb{R}[D_8]^{(i+j) \bmod 2}$, $i, j = 0, 1$, since $b^{-1}ab = a^3$.

Example 20

Let $Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1}\rangle$, $H = \langle a \mid a^4 = 1\rangle \lhd G$, and

$$\mathbb{R}[Q_8]^{(0)} = \left\{x_0 1 + x_1 a + x_2 a^2 + x_3 a^3 \mid x_i \in \mathbb{R}\right\},$$
$$\mathbb{R}[Q_8]^{(1)} = \left\{y_0 b + y_1 ab + y_2 a^2 b + y_3 a^3 b \mid y_i \in \mathbb{R}\right\}.$$

So, $\mathbb{R}[Q_8]^{(i)}\mathbb{R}[Q_8]^{(j)} \subseteq \mathbb{R}[Q_8]^{(i+j) \bmod 2}$, $i, j = 0, 1$, since $b^{-1}ab = a^3$ and $a^2 = b^2$.

Ideal $J = (1+\tau)$ in $\mathbb{R}[D_8]$ and $\mathbb{R}[Q_8]$ is homogeneous

Example 21

Let $J = (1+\tau) \subset \mathbb{R}[D_8] = \mathbb{R}[D_8]^{(0)} \oplus \mathbb{R}[D_8]^{(1)}$ where $\tau = a^2 \in Z(D_8)$. Then, $1+\tau \in \mathbb{R}[D_8]^{(0)}$. Let $j = (X^{(0)} + X^{(1)})(1+\tau) \in J$ where $X^{(i)} \in \mathbb{R}[D_8]^{(i)}$, $i = 0, 1$. Then, J is homogeneous because the homogeneous parts of j belong to J:

$$J \ni j = (X^{(0)} + X^{(1)})(1+\tau) = \underbrace{X^{(0)}(1+\tau)}_{j^{(0)} \in J} + \underbrace{X^{(1)}(1+\tau)}_{j^{(1)} \in J}.$$

Example 22

Let $J = (1+\tau) \subset \mathbb{R}[Q_8] = \mathbb{R}[Q_8]^{(0)} \oplus \mathbb{R}[Q_8]^{(1)}$ where $\tau = a^2 \in Z(Q_8)$. Then, $1+\tau \in \mathbb{R}[Q_8]^{(0)}$. Let $j = (X^{(0)} + X^{(1)})(1+\tau) \in J$ where $X^{(i)} \in \mathbb{R}[Q_8]^{(i)}$, $i = 0, 1$. Then, J is homogeneous because the homogeneous parts of j belong to J:

$$J \ni j = (X^{(0)} + X^{(1)})(1+\tau) = \underbrace{X^{(0)}(1+\tau)}_{j^{(0)} \in J} + \underbrace{X^{(1)}(1+\tau)}_{j^{(1)} \in J}.$$

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Conclusions and questions

(a) Every Clifford algebra $C\ell_{p,q}$ is isomorphic to one of the quotient algebras:
   (i) $\mathbb{R}[N_{\mathrm{even}}]/\mathcal{J}$, $\mathbb{R}[N_{\mathrm{odd}}]/\mathcal{J}$, $\mathbb{R}[S_k]/\mathcal{J}$ when $C\ell_{p,q}$ is simple, and
   (ii) $\mathbb{R}[\Omega_{\mathrm{even}}]/\mathcal{J}$, $\mathbb{R}[\Omega_{\mathrm{odd}}]/\mathcal{J}$ when $C\ell_{p,q}$ is semisimple,

   of Salingaros vee groups N_{even}, N_{odd}, S_k, Ω_{even}, and Ω_{odd} modulo the ideal $J = (1+\tau)$ for a central element τ of order 2.

(b) Is there a $\mathbb{Z}_2$-graded isomorphism between $\mathbb{R}[G_{p,q}]/\mathcal{J}$ and $C\ell_{p,q}$?

(c) Should $J = (1+\tau)$ always be homogeneous?

(d) How does the group structure of $G_{p,q}$, e.g., presence of normal subgroups and the central product structure, carry over to the algebra structure of $C\ell_{p,q}$? If so, how?

(e) Use the central-product structure of Salingaros vee groups $G_{p,q}$ to explain the $\mathbb{Z}_2$-gradation of $\mathbb{R}[G_{p,q}]/\mathcal{J}$, and so of $C\ell_{p,q}$.

(f) Apply the character theory and real representation methods of 2-groups to the group algebras $\mathbb{R}[G_{p,q}]$ and their quotients $\mathbb{R}[G_{p,q}]/\mathcal{J}$, and hence to the Clifford algebras $C\ell_{p,q}$.

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