Problem Set #3

Risk-free investment. The bond price of unit share is represented by a continuously compounded return \( B(t) = e^{rt} \). It is a solution to the ordinary differential equation \( \frac{dB}{dt} = rB \), or equivalently we can write

\[
dB = rB dt; \quad B(0) = 1
\]

with an initial value.

Stock price. By introducing a volatility term \( \sigma X dW \), the price \( X(t) \) of a stock at time \( t \) evolves according to the stochastic differential equation (SDE)

\[
dX = \mu X dt + \sigma X dW; \quad X(0) = P_0
\]

with the initial value \( P_0 \). Here \( \mu X dt \) is called a drift term.

Portfolio and self-financing. We introduce a pair \( \phi(X(t), t) \) and \( \psi(X(t), t) \) of processes with unknown smooth functions \( \phi(x, t) \) and \( \psi(x, t) \), and construct the portfolio

\[
V(t) = \phi(X(t), t)X(t) + \psi(X(t), t)B(t).
\]

Here the change \( dV \) in the value of the portfolio should depend only on the changes \( dX \) and \( dB \). Then the self-financing assumes that

\[
dV = \phi dX + \psi dB
\]

Strike price and terminal condition. European call option is a financial contract between the buyer and the seller, in which the buyer has the right to buy one share of a particular stock at the strike price \( k \) at a certain time \( T \). Then the “proper” portfolio \( V(t) \) must meet the terminal condition

\[
V(T) = [X(T) - k]_+
\]

in order to fulfill the obligation to pay the difference (payoff) if the stock price \( X(T) \) is higher than the strike price \( k \). Here we define \( [x]_+ = x \) for \( x \geq 0 \); \( [x]_+ = 0 \) for \( x < 0 \), as in Problem Set #1.

Option pricing. By completing Problems 1–4, we will find the explicit formula for the proper portfolio \( V(t) \) in terms of \( X(t) \) satisfying (3.1)–(3.5). In particular we are interested in calculating \( V(0) \) by means of the initial value \( P_0 \) and the strike price \( k \), which is now known as the Black-Scholes formula.

Problem 1. We will find \( X(t), 0 \leq t \leq T \), by solving the SDE (3.2), and calculate \( E[V(T)] \).

(a) Use \( u(x, t) = \ln x \) in order to apply Ito formula, and solve the SDE (3.2).

(b) Express \( X(T) = e^{w(T) + \gamma(T)} \), and find \( \gamma(T) \) in terms of \( P_0, \mu, \sigma \) and \( T \).
(c) Calculate $E[V(T)]$, and express it in terms of $\Phi(x)$.

Hint: (i) $\sigma W(T) + \gamma(T)$ is distributed as $N(\gamma(T), \sigma^2 T)$. (ii) Apply the result from Problem Set #1.

**Problem 2.** Equation (3.3) implies that $V(t)$ is a function of $X(t)$. Thus, we write

$$V(t) = u(X(t), t), \quad 0 \leq t \leq T$$

(3.6)

with a smooth function $u(x, t)$. But the choice of $u(x, t)$ is not determined yet.

(a) Use Ito formula to (3.2) and (3.6), and obtain

$$dV = \left( u_t + \mu Xu_x + \frac{\sigma^2}{2} X^2 u_{xx} \right) dt + \sigma Xu_x dW$$

(3.7)

where $u_t = \frac{\partial}{\partial t} u(X(t), t)$, $u_x = \frac{\partial}{\partial x} u(X(t), t)$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u(X(t), t)$.

(b) Express $dV$ in a different way using (3.1), (3.2) and (3.4). Then compare it with (3.7) and show that

$$\left( u_t + \mu Xu_x + \frac{\sigma^2}{2} X^2 u_{xx} \right) dt = (\mu \phi X + r \psi B) dt$$

$$\sigma Xu_x dW = \sigma \phi X dW$$

(3.8) \hspace{1cm} (3.9)

(c) Equation (3.9) implies $u_x = \phi$. Thus, Equation (3.8) becomes

$$\left( u_t + \frac{\sigma^2}{2} X^2 u_{xx} \right) dt = r \psi B dt$$

(3.10)

Using (3.3), (3.6) and (3.10), show that

$$\left( u_t + r Xu_x + \frac{\sigma^2}{2} X^2 u_{xx} - ru \right) dt = 0$$

(3.11)

where $u = u(X(t), t)$ as introduced in (3.6).

**Problem 3.** Equation (3.11) implies that if $u(x, t)$ is the solution to the partial differential equation (PDE)

$$u_t + r Xu_x + \frac{\sigma^2}{2} x^2 u_{xx} - ru = 0$$

(3.12)

then $V(t) = u(X(t), t)$ satisfies (3.1)–(3.5). Here we have $u_t = \frac{\partial}{\partial t} u(x, t)$, $u_x = \frac{\partial}{\partial x} u(x, t)$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u(x, t)$. And the terminal condition (3.5) becomes

$$u(x, T) = [x - k]_+$$

(3.13)
(a) Let \( v(x,t) = e^{-rt}u(x,t) \). Show that

\[
 u_t = v_t e^{rt} + rve^{rt}; \quad u_x = v_x e^{rt}; \quad u_{xx} = v_{xx} e^{rt}
\]

where \( v_t = \frac{\partial}{\partial t} v(x,t) \), \( v_x = \frac{\partial}{\partial x} v(x,t) \) and \( v_{xx} = \frac{\partial^2}{\partial x^2} v(x,t) \). 

(b) Using (3.14), show that the PDE (3.12) with (3.13) is equivalent to

\[
 v_t + r xv_x + \frac{\sigma^2}{2} x^2 v_{xx} = 0; \quad v(x,T) = e^{-rT}[x - k]_+
\]

(c) Introduce the change of variables by \( z = \ln x + \left( r - \frac{\sigma^2}{2} \right)(T - t) \) and \( s = T - t \). And accordingly we can change \( p(z,s) = v(e^{z-(r-\sigma^2/2)s},T-s) \). Then show the following relations

\[
 v_t = -ps - \left( r - \frac{\sigma^2}{2} \right) p_z; \quad xv_x = p_z; \quad x^2 v_{xx} = p_{zz} - p_z
\]

(d) Using (3.16), show that the PDE (3.15) is equivalent to the heat equation

\[
 -ps + \frac{\sigma^2}{2} p_{zz} = 0; \quad p(z,0) = e^{-rT}[e^z - k]_+
\]

**Problem 4.** Continue from Problem 3. Now we can find \( u(x,t) \) satisfying the PDE (3.12), so that we can express \( V(t) \) in terms of \( X(t) \).

(a) Show that \( u(x,t) \) is related to \( p(z,s) \) via

\[
 u(x,t) = e^{rt}v(x,t) = e^{rt} p \left( \ln x + \left( r - \frac{\sigma^2}{2} \right)(T - t), T - t \right)
\]

(b) Find the solution \( p(z,s) \) to the PDE (3.17), and express it in terms of the standard normal distribution \( \Phi(x) \).

(c) Find the solution \( u(x,t) \) and express it in terms of \( \Phi(x) \).

(d) Calculate the “proper” price \( V(0) \) for the option contract, and confirm the following statement:

“As reasonable as this may all seem, \( e^{-rT}E[V(T)] \) is in fact not the proper price.”

That is, show that

\[
 V(0) \neq e^{-rT}E[V(T)]
\]