Problem Set #2

Stochastic differential. Let $W(t)$ be a Wiener process, and let $X(t)$, $F(t)$ and $G(t)$ be progressively measurable stochastic processes. Then the stochastic differential
d\begin{equation}
X(t) - X(0) = \int_0^t dX(s) = \int_0^t F(s) \, ds + \int_0^t G(s) \, dW(s)
\end{equation}
is defined as

where $\int_0^t G(s) \, dW(s)$ is Ito integral.

Ito formula. Let $u(x,t)$ be a smooth function (i.e., differentiable enough). Here we can introduce a stochastic process

\begin{equation}
Y(t) = u(X(t), t)
\end{equation}

Then $Y(t)$ has the stochastic differential
d\begin{equation}
dY = u_t \, dt + u_x \, dX + \frac{1}{2} u_{xx} G^2 \, dt
\end{equation}

where

\begin{align*}
u_t &= \frac{\partial}{\partial t} u(X(t), t); \quad u_x = \frac{\partial}{\partial x} u(X(t), t); \quad u_{xx} = \frac{\partial^2}{\partial x^2} u(X(t), t)
\end{align*}

Separable differential equation. The ordinary differential equation
d\begin{equation}
dv/dt = vg(t); \quad v(0) = 1
\end{equation}
is called “separable,” and it has the solution
\begin{equation}
v(t) = \exp \left( \int_0^t g(s) \, ds \right)
\end{equation}

Problem 1. Let $F \equiv 0$ and $G \equiv 1$ in (2.1), and let $u(x, t) = x^n$ in (2.2). Then answer the following questions.

1. Show that $dY = nW^{n-1} \, dW + \frac{1}{2} n(n-1) W^{n-2} \, dt$.

2. Show that $\int_0^t W(s)^{n-1} \, dW(s) = \frac{1}{n} W(t)^n - \frac{n-1}{2} \int_0^t W(s)^{n-2} \, ds$. 

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Problem 2. Let $F \equiv 0$ and $G \equiv 1$ in (2.1), and let $u(x, t) = tx$ in (2.2). Then answer the following questions.

1. Show that $dY = W dt + t dW$.
2. Show that $\int_0^t s dW(s) = t W(t) - \int_0^t W(s) ds$.

Problem 3. Answer the following questions to solve the stochastic differential equation

$$dX = e^{-t} X dW; \quad X(0) = 1 \quad (2.5)$$

1. Let $F \equiv 0$ and $G(t) = e^{-t} X(t)$ in (2.1), and let $u(x, t) = \ln(x)$ in (2.2). Then show that $dY = e^{-t} dW - \frac{e^{-2t}}{2} dt$.
2. Show that $Y(t) = \int_0^t e^{-s} dW(s) - \frac{1}{4} (1 - e^{-2t})$, and find the solution to (2.5)

Problem 4. Answer the following questions to solve the stochastic differential equation

$$dX = (1 + \lambda X) dt + \sigma^2 dW; \quad X(0) = 0 \quad (2.6)$$

where $\lambda$ and $\sigma^2$ are constant values.

1. Let $F = 1 + \lambda X$ and $G(t) \equiv \sigma^2$ in (2.1), and let $u(x, t) = xe^{-\lambda t}$ in (2.2). Then show that $dY = e^{-\lambda t} dt + \sigma^2 e^{-\lambda t} dW$.
2. Find the solution to (2.6)

Problem 5. Answer the following questions to solve the stochastic differential equation

$$dX = (1 + t X) dt + \sigma^2 dW; \quad X(0) = 0 \quad (2.7)$$

where $\sigma^2$ is a constant value.

1. Suppose that $v(t)$ is a smooth function not yet determined, and that $v'(t)$ is the derivative function of $v(t)$. Let $F = 1 + t X$ and $G(t) \equiv \sigma^2$ in (2.1), and let $u(x, t) = \frac{x}{v(t)}$ in (2.2). Then show that $dY = \left(-\frac{v'}{v^2} + \frac{t}{v}\right) X dt + \frac{1}{v} dt + \frac{\sigma^2}{v} dW$.
2. Find the function $v(t)$ by solving the separable differential equation $v' = tv$ with $v(0) = 1$.
3. Show $dY = e^{-t^2/2} dt + \sigma^2 e^{-t^2/2} dW$, and find the solution to (2.7)