2.23 Theorem. *Let A and B be sets in a metric space. If $A \subset B$, then $L(A) \subset L(B)$.*

2.24 Theorem. *Let A and B be sets in a metric space. Then $L(A \cup B) = L(A) \cup L(B)$.*

It is worth noting that you can't replace union with intersection in 2.24. Just consider $\mathbb{Q}$ and $\mathbb{P}$ in the space $\mathbb{R}$: $\mathbb{Q} \cap \mathbb{P} = \varnothing$, but $L(\mathbb{Q}) \cap L(\mathbb{P}) = \mathbb{R}$ since $L(\mathbb{Q}) = L(\mathbb{P}) = \mathbb{R}$.

§ 3 *Subspaces*

The notion of a subspace seems very simple, at first. It is similar to the notion of a vector subspace from linear algebra or a subgroup from abstract algebra—except that there are absolutely no requirements at all for a subset to be a subspace.

3.1 Theorem. *Let $\langle X, d\rangle$ be a metric space, and let $Y \subset X$. Let d_Y be the restriction of d to $Y \times Y$:*

$$d_Y(x, y) = d(x, y) \quad \textit{for } x, y \in Y$$

Then $\langle Y, d_Y\rangle$ is a metric space. Any such space is called a subspace of $\langle X, d\rangle$

We usually abuse notation in several ways. We'll just say "Y is a subspace of X" when the metrics are clear from the context. We usually use the same name (i.e., just d) for the metric d_Y of the subspace. This is almost never ambiguous since the metrics are equal whenever they are both defined.

However, subspaces can be a bit more confusing than they seem to be at first glance. For example, consider the "half closed" interval $[0, 1)$ as a subspace of $\mathbb{R}$. Suppose we now let $U = [0, \frac{1}{3})$. Then U is both a subset of $\mathbb{R}$ and a subset of the subspace $[0, 1)$. As a subset of $\mathbb{R}$, U is neither open nor closed. But, when considered as a set in the subspace $[0, 1)$, U is an open set. To emphasize this, we sometimes refer to a set which is open in a subspace as a *relatively open* set of the subspace. We similarly refer to *relatively closed* sets. Other definitions (limit point, closure, etc) can also be considered in either the "parent" space or a subspace. When needed, we use a subscript to indicate the space we are "working" in. For example: if we let $X = [0, 2)$ and $A = (1, 2)$, then $\mathrm{cl}_{\mathbb{R}}(A) = [1, 2]$, but $cl_X(A) = [1, 2)$.

3.2 Theorem. *Let Y be a subspace of a metric space X, and let $A \subset Y$. Then A is relatively open in Y iff $A = Y \cap U$ for some open set U of X. Similarly, A is relatively closed in Y iff $A = Y \cap F$ for some closed set F of X.*

3.3 Theorem. *Let Y be a subspace of a metric space X. If Y is also an open (closed) subset of X, then a subset of Y is relatively open (closed) in Y iff it is open (closed) in X.*

3.4 Theorem. *Let Y be a subspace of a metric space X, and let $A \subset Y$. Then $L_Y(A) = L_X(A) \cap Y$ and $\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap X$.*

3.5 Proposition. *Let Y be a subspace of a metric space X, and let $A \subset Y$. Then $\mathrm{int}_Y(A) = \mathrm{int}_X(A) \cap Y$, $\mathrm{ext}_Y(A) = \mathrm{ext}_X(A) \cap Y$, and $\mathrm{bd}_Y(A) = \mathrm{bd}_X(A) \cap Y$.*

§ 4 *Connected Sets and Spaces*

Connectedness is an idea that seems simple and straightforward. However, it turns out to rather tricky to give it a precise definition.

4.1 Definition. Let A and B be subsets of a metric space. Then A and B are *mutually separated* means that neither A nor B contains a point or limit point of the other. A *separation* of a set C is a pair $\{A, B\}$ of mutually separated separated sets such that $C = A \cup B$ and neither A nor B is empty. A set is *disconnected* means that there is a separation of it. A set is connected means that it is not disconnected.

It is usually easier to show that a set is disconnected than to show that it connected, since it is generally easier to find a separation than to show that no separation exists. For example, in $\mathbb{R}$, the set $C = (0, 1) \cup (2, 3)$ is disconnected because $\{(0, 1), (2, 3)\}$ is a separation. It is harder to prove that each of the intervals $(0, 1)$ and $(2, 3)$ are themselves connected (4.xxx, below). Note that the sets of a separation can have a limit point in common, i.e., $(0, 1)$ and $(1, 2)$ are mutually separated (in $\mathbb{R}$). We can also separate $\mathbb{Q}$ by considering, for example, $\mathbb{Q} \cap (-\infty, \sqrt{2})$ and $\mathbb{Q} \cap (\sqrt{2}, \infty)$.

Some other things to note: A and B are mutually separated means that $A \cap B$, $A \cap L(B)$, and $L(A) \cap B$ are all empty. This can also be written as $A \cap \mathrm{cl}(B) = \varnothing$, and $\mathrm{cl}(A) \cap B = \varnothing$ if that is more convenient to use (but $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$ is too much to require). Finally, note that for any set A, A and $\varnothing$ satisfy the "mutually separated" definition. So, it is important that the definition of a separation requires the sets to be non-empty. Otherwise, there wouldn't be any connected sets at all. This also implies that any set with just one point is connected, and that $\varnothing$ is also connected, since the union of any separation must have at least two points.

We say that a metric space X is connected if it is a connected subset of itself, i.e., there is no separation $\{A, B\}$ of X.

4.2 Theorem. *Let X be a metric space, and let A and B be disjoint subsets of X whose union is X. The following are equivalent:*

(i) *A and B are mutually separated.*

(ii) *A and B are both open sets.*

(iii) *A and B are both closed sets.*

This leads to the following clever-sounding acronym.

4.3 Definition. A subset of a metric space is *clopen* means that it is both open and closed.

4.4 Theorem. *A metric space X is connected iff the only clopen subsets of X are $\varnothing$ and X.*

A relatively nice fact is that we *don't* ever need to use the term "relatively connected" for a set in a subspace of a metric space.

4.5 Theorem. *Let X be a metric space, and $Y \subset X$. The following are equivalent:*

(i) *Y is a connected subset of X.*

(ii) *The only relatively clopen subsets of Y are $\varnothing$ and Y.*

(iii) *The subspace Y of X is a connected metric space.*

4.6 Theorem. *$\mathbb{R}$ is a connected space.*

4.7 Theorem. *A subset of $\mathbb{R}$ is connected iff it is an interval, i.e.: (a,b), $[a,b]$, $(a,b]$, $[a,b)$, $[a,\infty)$, (a,∞), $(-\infty,a]$, $(-\infty,a)$, or $(-\infty,\infty)$.*

4.8 Theorem. *$\mathbb{R}^2$ is a connected space.*

Note that the $(0,1) \cup (2,3)$ example shows that the union of connected sets can be disconnected.

4.9 Proposition. *If A and B are connected sets then $A \cap B$ is connected.*

4.10 Theorem. *If A and B are connected sets and $A \cap B \neq \varnothing$ then $A \cup B$ is connected.*

4.11 Theorem. *If* A *is connected, then* $\mathrm{cl}(A)$ *is connected.*

4.12 Proposition. *If* $A_1, A_2, A_3, \ldots$ *is an infinite sequence of connected sets such that* $A_1 \supset A_2 \supset A_3 \supset \ldots$, *then* $\bigcap_{i=1}^{\infty} A_i$ *is connected.*

4.13 Proposition. *If* A *is a connected subset of* $\mathbb{R}^2$, *then* $\mathrm{bd}(A)$ *is connected.*

4.14 Proposition. *If* A *is a non-trivial connected set, then* $A \subset L(A)$.

4.15 Theorem. *Let* X *be a metric space, and let* $\sim$ *be the relation on the points of* X *defined by:* $a \sim b$ *iff there is a connected subset of* X *that contains both* a *and* b. *Then* $\sim$ *is an equivalence relation.*

4.16 Definition. The equivalence classes of 4.15 are called the *connected components* of a space. A space is *totally disconnected* means that each of its connected components contains just a single point.

4.17 Theorem. $\mathbb{Q}$ *and* $\mathbb{P}$ *are totally disconnected spaces.*

4.18 Theorem. *Connected components are themselves connected sets.*

4.19 Proposition. *Connected components must be closed sets.*

4.20 Proposition. *Connected components must be open sets.*

§ 5 *Sets, Relations and Functions*

A *set* is a collection of "objects". No further *definition* of set is really possible, since "set" is usually considered the basic undefined notion of modern mathematics. Instead of defining what a set is, one can study axiom systems that sets should satisfy, much in the same way that Euclid studied axioms for geometry as opposed to *defining* what points and angles are. For now, we simply need some terms and definitions about sets. This treatment is often referred to as "naive" set theory. For clarity, we'll repeat some of the definitions and notation that have already been introduced before expanding on them.

The objects which make up a set are called the *elements* of the set. The notation $x \in X$ means that x is an element of the set X. If X and Y are sets, then $X \subset Y$ is read "X is a *subset* of Y", and simply means that every element of X is an element of Y. Note that every set is a subset of itself. If $X \subset Y$ and $X \neq Y$, we write $X \subsetneq Y$, and say that X is a *proper* subset of Y. The notation $X \subseteq Y$ is synonymous with $X \subset Y$, but is sometimes used to emphasize that X might in fact be equal to Y. (In olden days, some authors used $\subset$ for $\subsetneq$ in order to ease the burden on typesetters.) A fundamental fact (actually an axiom) is that if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$. This just says that a set is determined by its elements.

Curly braces are usually used to indicate sets. The notation $\{0, 1, 2, 4\}$ stands for the set whose elements are exactly the integers 0, 1, 2, and 4. The notation $\{x : \varphi(x)\}$ stands for the set of all objects which satisfy the "predicate" φ. For example,

$$\{x : x \text{ is a real number and } x > 0 \text{ and } x < 1\}$$

defines the open interval $(0, 1)$. (For now, we won't worry about exactly what a "predicate" is; we'll just use common sense.) A useful abbreviation is $\{x \in A : \varphi(x)\}$ which just stands for $\{x : x \in A \text{ and } \varphi(x)\}$. Simple but important definitions are:

$$\begin{aligned} X \cap Y &\stackrel{\text{def}}{=} \{a : a \in X \text{ and } a \in Y\} \\ X \cup Y &\stackrel{\text{def}}{=} \{a : a \in X \text{ or } a \in Y\} \\ X \setminus Y = X - Y &\stackrel{\text{def}}{=} \{a : a \in X \text{ and } a \notin Y\} \end{aligned}$$

Note that the complement of set (e.g., in 1.16) is just a special case of subtraction, where the set X is understood to be the "whole space."

You are already familiar with some mathematical sets: the set $\mathbb{R}$ of real numbers, the set $\mathbb{Z}$ of integers, the set $\mathbb{Q}$ of rational numbers, and also the empty set $\varnothing$ which is the unique set that has no elements in it. We also use $\mathbb{P}$ for the set of irrational numbers, $\mathbb{N}$ for the set of positive integers ($\{1, 2, \dots\}$) and ω for the set of non-negative integers ($\{0, 1, 2 \dots\}$). Note that some authors use $\mathbb{N}$ for the non-negative integers, so you should be careful when reading books or papers if the distinction is important.

While we won't go into (for now) exactly what sort of objects can and can't be elements of sets, it is important to realize that sets can themselves serve as elements of other sets. For example, define some open intervals in the real line by the definition

$$I_n = (n, n+1) = \{ x \in \mathbb{R} : n < x < n+1 \}$$

where n is an integer. Then we can form sets such as $\{ I_0, I_1, I_2, I_4 \}$ or even $\{ I_n : n \in \mathbb{N} \}$. The term *collection* is synonymous with set, but it is often used because it is less confusing to think of a "collection of sets" instead of a "set of sets". Of course, once we have collections of sets, we can form collections of collections, etc, so the distinction between set and collection only helps for "one level". As an example, you should convince yourself that $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\varnothing, \{\varnothing\}\}$, and $\{\{\{\varnothing\}\}\}$ are all *different* objects.

We have already introduced the generalizations of intersection and union to an infinite sequence of sets. This generalizes further to the concept of *arbitrary* unions and intersections.

5.1 Definition. If $\mathcal{C}$ is any collection of sets (i.e., a set each of whose elements is itself a set), then

$$\bigcap \mathcal{C} \stackrel{\text{def}}{=} \{ x : x \in X \text{ for every } X \in \mathcal{C} \}$$

$$\bigcup \mathcal{C} \stackrel{\text{def}}{=} \{ x : x \in X \text{ for some } X \in \mathcal{C} \}$$

It is easy to see that this generalizes 1.5, since $\bigcap_{i=0}^{\infty} X_i = \bigcap\{ X_0, X_1, \dots \}$ and $\bigcup_{i=0}^{\infty} X_i = \bigcup\{ X_0, X_1, \dots \}$. For a trivial example which is not a sequence, let $\mathcal{C}$ denote the collection of all open intervals in $\mathbb{R}$ which contain 0. Then $\bigcap \mathcal{C} = \{0\}$ and $\bigcup \mathcal{C} = \mathbb{R}$.

A key fact about open and closed sets is the following, which generalizes the true parts of 2.7.

5.2 Theorem. *Let $\mathcal{C}$ be a collection of open sets in a metric space. Then $\bigcup \mathcal{C}$ is an open set. Let $\mathcal{C}$ be a non-empty collection of closed sets in a metric space. Then $\bigcap \mathcal{C}$ is a closed set.*

It is important in 5.2 that we take the empty set to be (vacuously) both open and closed. It is also worth noting that $\bigcup \varnothing = \varnothing$, but that this won't be true for $\bigcap \varnothing$.

The unordered pair containing the elements x and y is simply the set $\{x, y\}$. The *ordered pair* $\langle x, y \rangle$ is usually thought of as the elements x and y with x "first" and y "second". This can be given a precise definition as follows:

5.3 Definition.

$$\langle x, y \rangle \stackrel{\text{def}}{=} \{\, \{x\},\ \{x, y\} \,\}$$

In order to see that this definition captures the idea of ordered pair, it is only necessary to prove the following.

5.4 Theorem. *For any x, y, x', and y', $\langle x, y \rangle = \langle x', y' \rangle$ if, and only if, $x = x'$ and $y = y'$.*

5.5 Proposition. *If we throw out 5.3 and instead adopt the alternate definition $\langle x, y \rangle = \{\, x,\ \{y\} \,\}$, then Theorem 5.2 will still hold.*

Another common notation for ordered pair is (x, y). The advantage to using angle brackets instead of parenthesis is that we will avoid confusing pairs and intervals, i.e., $\langle 0, 1 \rangle$ is the ordered pair and $(0, 1)$ is the open interval $\{\, x \in \mathbb{R} : 0 < x < 1 \,\}$.

A *relation* is a set of ordered pairs. If R is a relation then

$$\begin{aligned} \operatorname{dom}(R) &\stackrel{\text{def}}{=} \{\, x : \text{there is some } y \text{ such that } \langle x, y \rangle \in R \,\} \\ \operatorname{ran}(R) &\stackrel{\text{def}}{=} \{\, y : \text{there is some } x \text{ such that } \langle x, y \rangle \in R \,\} \end{aligned}$$

The sets $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are called the *domain* and *range* of the relation R. The relation thus tells us which elements of the domain are "related" to which elements of the range. The largest possible relation between the sets X and Y is:

5.6 Definition.

$$X \times Y \stackrel{\text{def}}{=} \{\, \langle x, y \rangle : x \in X \text{ and } y \in Y \,\}$$

This is called the *Cartesian product* of X and Y after René Descartes.

We usually think of a function as a rule or assignment of an element of the range to each element of the domain. This can be made more precise by defining a function to be a relation which relates exactly one element of its range to each element of its domain.

5.7 Definition. A *function* is a relation f such that for each $x \in \operatorname{dom}(f)$ there is exactly one $y \in \operatorname{ran}(f)$ such that $\langle x, y\rangle \in f$.

This definition makes a function into a concrete object (a set) instead of an abstract rule. In order to emphasize that we *think* of functions as rules, we use the notation $f(x) = y$ as an abbreviation for the statement $\langle x, y\rangle \in f$. We also use the notation $f : X \to Y$ in order to introduce functions. This is read "f is a function from X to Y", and its meaning is: f is a function, $\operatorname{dom}(f) = X$, and $\operatorname{ran}(f) \subset Y$. Only requiring that $\operatorname{ran}(f)$ be a subset of Y gives us the freedom to write such things as "define $f : \mathbb{R} \to \mathbb{R}$ by the equation $f(x) = x^2$", which really just shorthand for $f = \{\, \langle x, y\rangle : x, y \in \mathbb{R} \text{ and } y = x^2 \,\}$ (note that $\operatorname{ran}(f) = [0,\infty))$. If Y is equal to $\operatorname{ran}(f)$ we say that f is a function from X *onto* Y. Note that this way of looking at a function makes it clear that a function is more than an equation. If we define $g\colon (0,1) \to \mathbb{R}$ by $g(x) = x^2$, then $g \neq f$ even though the "equations" for f and g are the same. In fact, $g \subsetneq f$! On the other hand, if we define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = (x+1)^2 - 2x - 1$ then (as sets) $h = f$. A useful exercise in understanding this definition of function is to look at inverses and compositions.

5.8 Definitions. If R is a relation, then the *inverse* of R is defined by

$$R^{-1} \overset{\text{def}}{=} \{\, \langle y, x\rangle : \langle x, y\rangle \in R \,\}.$$

and if R_1 and R_2 are relations, then their *composition* is defined by

$$R_2 \circ R_1 \overset{\text{def}}{=} \{\, \langle x, z\rangle : \text{there exists some } y \text{ such that } \langle x, y\rangle \in R_1 \text{ and } \langle y, z\rangle \in R_2 \,\}.$$

(The order is "reversed" to make the definition agree with the standard $f \circ g(x) = f(g(x))$ equation).

A function f is 1:1 means that no two members of f can have the same second element and different first elements, i.e., if $x, y \in \operatorname{dom}(f)$ and $x \neq y$, then $f(x) \neq f(y)$. If f is a function and $A \subset \operatorname{dom}(f)$, then $f\restriction_A \overset{\text{def}}{=} f \cap (A \times \operatorname{ran}(f))$ is called the *restriction* of f to A. Its range $\operatorname{ran}(f\restriction_A) = \{\, f(a) : a \in A \,\}$ is called the *image* of A under f and is denoted by $f(A)$. The *inverse image* under f of a set A is the set $f^{-1}(A) \overset{\text{def}}{=} \{\, x \in \operatorname{dom}(f) : f(x) \in A \,\}$.

5.9 Theorem. *Let f be a function. The following statements are equivalent:*

(1) *f is 1:1.*

(2) *f^{-1} is a function.*

(3) *There is a function g such that $g \circ f = \mathrm{id}_{\mathrm{dom}(f)}$ (the function $\mathrm{id}_X \stackrel{\text{def}}{=} \{\, \langle x, x \rangle : x \in X \,\}$ is the identity function on the set X).*

Note that $f^{-1}(A)$ for a set A might mean two different things. Fortunately, they are the same set.

5.10 Theorem. *If f is 1:1 and $A \subset \mathrm{ran}(f)$, then the image of A under the function f^{-1} is equal to inverse image of A under f*

5.11 Definitions. A *1:1 correspondence* between sets X and Y is a 1:1 function f from X onto Y (i.e., $X = \mathrm{dom}(f)$ and $Y = \mathrm{ran}(f)$). When such an f exists, then f^{-1} is a 1:1 correspondence between Y and X, so we can simply say that X and Y are in 1:1 correspondence. We also say that X and Y have the "same cardinality," although this doesn't define exactly what the "cardinality" is as an object. A set is called *finite* if it is either empty or in 1:1 correspondence with a set of the form $\{1, \ldots, n\}$ where $n \in \mathbb{N}$. A set is *infinite* if it is not finite. A set is called *countably infinite* if it is in 1:1 correspondence with $\mathbb{N}$. A set is called *uncountable* if it is neither finite nor countably infinite.

5.12 Theorem (Cantor). *The sets ω, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite.*

5.13 Theorem (Cantor). *The set $\mathbb{R}$ is uncountable.*

5.14 Definition. If X is any set, then $\mathcal{P}(x) \stackrel{\text{def}}{=} \{\, A : A \subset X \,\}$ is called the power set of X. It contains all the subsets of X, including $\varnothing$ and X itself.

5.15 Theorem (Cantor). *If X is any set, then there does not exist a 1:1 correspondence between X and $\mathcal{P}(X)$.*

5.16 Theorem. *There is a 1:1 correspondence between $\mathcal{P}(\omega)$ and $\mathbb{R}$.*

5.17 Theorem (Schröder-Bernstein). *Let X and Y be sets. If X is in 1:1 correspondence with a subset of Y and Y is in 1:1 correspondence with a subset of X, then X and Y are in 1:1 correspondence.*