Integration Techniques

No silver bullet. Here is the review of basic integration techniques, together with some of the challenging problems presented at *University of North Texas Integration Bee*. You will see that even computer algebra systems such as Maple and Mathematica are not tough enough to complete them. Nevertheless, you could outperform smart software with the basic techniques we have learned.

Derivatives and integrals of basic functions.

Functions	Derivatives	Indefinite integrals
Fundamental theorem of calculus	$\frac{d}{dx}F(x) = f(x)$	$\int f(x)\,dx = F(x) + C$
Power and logarithmic functions	$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
	$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
Exponential functions	$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
Trigonometric functions	$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
	$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
	$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} = \sec^2 x$	$\int \frac{1}{\cos^2 x}\,dx = \tan x + C$
	$\frac{d}{dx}\cot x = -\frac{1}{\sin^2 x} = -\csc^2 x$	$\int \frac{1}{\sin^2 x}\,dx = -\cot x + C$
Hyperbolic functions	$\frac{d}{dx}\sinh x = \cosh x$	$\int \cosh x\,dx = \sinh x + C$
	$\frac{d}{dx}\cosh x = \sinh x$	$\int \sinh x\,dx = \cosh x + C$
	$\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x}$	$\int \frac{1}{\cosh^2 x}\,dx = \tanh x + C$
	$\frac{d}{dx}\coth x = -\frac{1}{\sinh^2 x}$	$\int \frac{1}{\sinh^2 x}\,dx = -\coth x + C$
Inverse trigonometric functions	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	
	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
Inverse hyperbolic functions	$\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$	$\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + C$
	$\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$	$\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1} x + C$
	$\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$	$\int \frac{1}{1-x^2}\,dx = \tanh^{-1} x + C$

Formulas and identities of basic functions.

Logarithmic and Exponential functions	$\log_a x = \dfrac{\ln x}{\ln a}$	$a^x = e^{x \ln a}$
Trigonometric functions	$\sec x = \dfrac{1}{\cos x}$	$\csc x = \dfrac{1}{\sin x}$
	$\tan x = \dfrac{\sin x}{\cos x}$	$\cot x = \dfrac{\cos x}{\sin x}$
	$\cos^2 x + \sin^2 x = 1$	$1 - \cos^2 x = \sin^2 x$
	$1 + \tan^2 x = \sec^2 x$	$\sec^2 -1 = \tan^2 x$
	$\sin 2x = 2 \sin x \cos x$	$\sin x = 2 \sin \dfrac{x}{2} \cos \dfrac{x}{2} = \dfrac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ †
	$\cos 2x = \cos^2 x - \sin^2 x$	$\cos x = \cos^2 \dfrac{x}{2} - \sin^2 \dfrac{x}{2} = \dfrac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ †
	$\sin^2 x = \dfrac{1 - \cos 2x}{2}$	$\cos^2 x = \dfrac{1 + \cos 2x}{2}$
	$A \cos x + B \sin x = \sqrt{A^2 + B^2} \cos(x - \theta)$ where $A > 0$ and $\theta = \tan^{-1}(B/A)$.	
Hyperbolic functions	$\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\sinh^{-1} x = \ln \left\| x + \sqrt{x^2+1} \right\|$
	$\cosh x = \dfrac{e^x + e^{-x}}{2}$	$\cosh^{-1} x = \ln \left\| x + \sqrt{x^2-1} \right\|$
	$\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$	$\tanh^{-1} x = \dfrac{1}{2} \ln \left\| \dfrac{1+x}{1-x} \right\|$
	$\cosh^2 x - \sinh^2 x = 1$	

† You may find them useful in the following "half-angle tangent" substitution:

Substitution	Identities
$t = \tan\left(\dfrac{x}{2}\right), \quad \dfrac{x}{2} = \tan^{-1} t;$	$\sin x = \dfrac{2t}{1+t^2}, \quad \cos x = \dfrac{1-t^2}{1+t^2}, \quad dx = \dfrac{2}{1+t^2}\, dt.$

Example. $$\int \frac{1+\sin x}{1+\cos x}\, dx = \int \frac{(1+t)^2}{1+t^2}\, dt = \int \left(1 + \frac{2t}{1+t^2}\right) dt = t + \int \frac{1}{s}\, ds \qquad \boxed{\text{(A.1) } s = 1+t^2,\ \tfrac{ds}{dt} = 2t}$$

$$= t + \ln|s| + C = \tan\frac{x}{2} + \ln\left|1 + \tan^2 \frac{x}{2}\right| + C$$

Strategy for integration with sample problems.

A. Substitution rules.

(A.1) $\displaystyle\int y\frac{dt}{dx}\,dx = \int y\,dt$

(a) $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = 2\int \sin t\,dt$

$\displaystyle = -2\cos t + C = -2\cos\sqrt{x} + C$

$\boxed{\text{(A.1) } t=\sqrt{x},\ \frac{dt}{dx}=\frac{1}{2\sqrt{x}}}$

(b) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \cos t\,dt$

$\displaystyle = \sin t + C = \sin(\ln x) + C$

$\boxed{\text{(A.1) } t=\ln x,\ \frac{dt}{dx}=\frac{1}{x}}$

(c) $\displaystyle\int \frac{\sin x}{1+\sin x}\,dx = \int \frac{\sin x(1-\sin x)}{1-\sin^2 x}\,dx$

$\displaystyle = \int \frac{\sin x}{\cos^2 x}\,dx - \int \frac{\sin^2 x}{\cos^2 x}\,dx = -\int \frac{1}{t^2}\,dt - \int \left[\frac{1}{\cos^2 x} - 1\right]dx$

$\displaystyle = \frac{1}{t} - [\tan x - x] + C = \frac{1}{\cos x} - \tan x + x + C$

$\boxed{\text{(A.1) } t=\cos x,\ \frac{dt}{dx}=-\sin x}$

(d) $\displaystyle\int \frac{1}{\sqrt{1-x^2+\sin^{-1}x - x^2\sin^{-1}x}}\,dx$ (Maple cannot evaluate; Mathematica can.)

$\displaystyle = \int \frac{1}{\sqrt{(1-x^2)(1+\sin^{-1}x)}}\,dx = \int \frac{1}{\sqrt{t}}\,dt$

$\displaystyle = 2t^{\frac{1}{2}} + C = 2\sqrt{1+\sin^{-1}x} + C$

$\boxed{\text{(A.1) } t=1+\sin^{-1}x,\ \frac{dt}{dx}=\frac{1}{\sqrt{1-x^2}}}$

(e) $\displaystyle\int \frac{1}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{\cos(x-\frac{\pi}{4})}\,dx$ (Mathematica cannot evaluate; Maple can.)

$\displaystyle = \frac{1}{\sqrt{2}}\int \frac{\cos(x-\frac{\pi}{4})}{1-\sin^2(x-\frac{\pi}{4})}\,dx = \frac{1}{\sqrt{2}}\int \frac{1}{1-t^2}\,dt$

$\displaystyle = \frac{1}{\sqrt{2}}\tanh^{-1} t + C = \frac{1}{\sqrt{2}}\tanh^{-1}\sin\left(x-\frac{\pi}{4}\right) + C$

$\boxed{\text{(A.1) } t=\sin(x-\frac{\pi}{4}),\ \frac{dt}{dx}=\cos(x-\frac{\pi}{4})}$

(A.2) $\displaystyle\int y\,dx = \int y\,\frac{dx}{dt}\,dt$

(a) $\displaystyle\int x^2\sqrt{x+4}\,dx = \int (t^2-4)^2\,t\,2t\,dt$

$\displaystyle = 2\int (t^6 - 8t^4 + 16t^2)\,dt = 2\left(\frac{1}{7}t^7 - \frac{8}{5}t^5 + \frac{16}{3}t^3\right) + C$

$\displaystyle = \frac{2}{7}(x+4)^{\frac{7}{2}} - \frac{16}{5}(x+4)^{\frac{5}{2}} + \frac{32}{3}(x+4)^{\frac{3}{2}} + C$

$\boxed{\text{(A.2) } t^2=x+4,\ \frac{dx}{dt}=2t}$

(b) $\int \sqrt{1-e^x}\,dx = \int t\,\frac{2t}{t^2-1}\,dt$ $\quad\boxed{\text{(A.2) } t^2 = 1-e^x,\ \frac{dx}{dt} = \frac{2t}{t^2-1}}$

$$= 2\int \left[1 - \frac{1}{1-t^2}\right] dt = 2\left[t - \tanh^{-1} t\right] + C = 2\sqrt{1-e^x} - 2\tanh^{-1}\sqrt{1-e^x} + C$$

(c) $\int \frac{1}{x^4-16}\,dx = \frac{1}{8}\int \frac{1}{t^4-1}\,dt$ $\quad\boxed{\text{(A.2) } x = 2t,\ \frac{dx}{dt} = 2}$

$$= \frac{1}{16}\int \left[\frac{1}{t^2-1} - \frac{1}{t^2+1}\right] dx = \frac{1}{16}\left[-\tanh^{-1} t - \tan^{-1} t\right] + C$$

$$= -\frac{1}{16}\tanh^{-1}\left(\frac{x}{2}\right) - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C$$

(d) $\int \frac{x^2}{\sqrt{x-1}}\,dx = 2\int (t^2+1)^2\,dt$ $\quad\boxed{\text{(A.2) } t^2 = x-1,\ \frac{dx}{dt} = 2t}$

$$= 2\int [t^4 + 2t^2 + 1]\,dt = \frac{2}{5}t^5 + \frac{4}{3}t^3 + 2t + C = \frac{2}{5}(x-1)^{\frac{5}{2}} + \frac{4}{3}(x-1)^{\frac{3}{2}} + 2(x-1)^{\frac{1}{2}} + C$$

(A.3) $\int \frac{1}{y}\frac{dy}{dx}\,dx = \ln|y| + C$

(a) $\int \sec x\,dx = \int \sec x\,\frac{\sec x + \tan x}{\sec x + \tan x}\,dx$ $\quad\boxed{\text{(A.3) } y = \sec x + \tan x,\quad \frac{dy}{dx} = \sec x\tan x + \sec^2 x}$

$$= \ln|y| + C = \ln|\sec x + \tan x| + C$$

(A.4) Trigonometric substitution.

Substitution		Identities	
$x = a\sin t,$	$t = \sin^{-1}\left(\frac{x}{a}\right);$	$a^2 - x^2 = a^2\cos^2 t,$	$dx = a\cos t\,dt.$
$x = a\tan t$	$t = \tan^{-1}\left(\frac{x}{a}\right);$	$a^2 + x^2 = a^2\sec^2 t,$	$dx = a\sec^2 t\,dt.$
$x = a\sec t$	$t = \cos^{-1}\left(\frac{a}{x}\right);$	$x^2 - a^2 = a^2\tan^2 t,$	$dx = a\sec t\tan t\,dt.$

(a) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \int dt = t + C = \sin^{-1} x + C$ $\quad\boxed{\text{(A.4) } x = \sin t}$

(b) $\int \frac{1}{1-x^2}\,dx = \int \sec t\,dt$ $\quad\boxed{\text{(A.4) } x = \sin t}$

$$= \ln|\sec t + \tan t| + C = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \tanh^{-1} x + C$$

(c) $\int \frac{1}{1+x^2}\,dx = \int dt = t + C = \tan^{-1} x + C$ $\quad\boxed{\text{(A.4) } x = \tan t}$

(d) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \int \sec t\,dt$ $\quad\boxed{\text{(A.4) } x = \tan t}$

$$= \ln|\sec t + \tan t| + C = \ln\left|\sqrt{1+x^2} + x\right| = \sinh^{-1} x + C$$

(e) $\int \frac{1}{\sqrt{x^2-1}}\,dx = \int \sec t\,dt$ $\quad\boxed{\text{(A.4) } x = \sec t}$

$$= \ln|\sec t + \tan t| + C = \ln\left|x + \sqrt{x^2-1}\right| = \cosh^{-1} x + C$$

B. Integration by parts: $\int u\,dv = uv - \int v\,du$

(a) $\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$ $\quad\boxed{\text{(B) } u = \ln x,\ v = \tfrac{2}{3}x^{\frac{3}{2}}}$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$

(b) $\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx$ $\quad\boxed{\text{(B) } u = \sin x,\ v = e^x}$ (Maple cannot simplify.)

$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x\,dx\right]$ $\quad\boxed{\text{(B) } u = \cos x,\ v = e^x}$

$= e^x \sin x - e^x \cos x - \int e^x \sin x\,dx$, which implies $\int e^x \sin x\,dx = \frac{e^2}{2}(\sin x - \cos x)$.

(c) $\int x^3 e^{-x^2}\,dx = \frac{1}{2}\int te^{-t}\,dt$ $\quad\boxed{\text{(A.1) } t = x^2,\ \tfrac{dt}{dx} = 2x}$

$= \frac{1}{2}\left[-te^{-t} + \int e^{-t}\,dt\right]$ $\quad\boxed{\text{(B) } u = t,\ v = -e^{-t}}$

$= \frac{1}{2}\left[-te^{-t} + \int e^{-t}\,dt\right] = \frac{1}{2}\left[-te^{-t} - e^{-t}\right] + C = -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + C$

(d) $\int \frac{x}{2 + e^x + e^{-x}}\,dx = 4\int \frac{t}{(e^t + e^{-t})^2}\,dt$ $\quad\boxed{\text{(A.2) } x = 2t,\ \tfrac{dx}{dt} = 2}$

$= \int \frac{t}{\cosh^2 t}\,dt = t\tanh t - \int \tanh t\,dt$ $\quad\boxed{\text{(B) } u = t,\ v = \tanh t}$

$= t\tanh t - \int \frac{\sinh t}{\cosh t}\,dt = t\tanh t - \int \frac{1}{s}\,ds$ $\quad\boxed{\text{(A.1) } s = \cosh t,\ \tfrac{ds}{dt} = \sinh t}$

$= t\tanh t - \ln|s| + C = (x/2)\tanh(x/2) - \ln|\cosh(x/2)| + C$

(e) $\int x^3 \sin x^2\,dx = \frac{1}{2}\int t\sin t\,dt$ $\quad\boxed{\text{(A.1) } t = x^2,\ \tfrac{dt}{dx} = 2x}$

$= \frac{1}{2}\left[-t\cos t + \int \cos t\,dt\right]$ $\quad\boxed{\text{(B) } u = t,\ v = -\cos t}$

$= -\frac{t}{2}\cos t + \frac{1}{2}\sin t + C = -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2 + C$

(f) $\int \sin x \ln|\cos x|\,dx = \int -\ln|t|\,dt$ $\quad\boxed{\text{(A.1) } t = \cos x,\ \tfrac{dt}{dx} = -\sin x}$

$= -t\ln|t| + \int dt$ $\quad\boxed{\text{(B) } u = \ln|t|,\ v = -t}$

$= -t\ln|t| + t + C = -\cos x \ln|\cos x| + \cos x + C$

(g) $$\int \frac{x^2}{(x^2+8)^{\frac{3}{2}}}\,dx = -x(x^2+8)^{\frac{1}{2}} + \int \frac{1}{(x^2+8)^{\frac{1}{2}}}\,dx \qquad \boxed{\text{(B) } u = x,\ v = -(x^2+8)^{\frac{1}{2}}}$$

$$= -\frac{x}{\sqrt{x^2+8}} + \int \frac{1}{\sqrt{t^2+1}}\,dt \qquad \boxed{\text{(A.2) } x = \sqrt{8}t,\ \tfrac{dx}{dt} = \sqrt{8}}$$

$$= -\frac{x}{\sqrt{x^2+8}} + \sinh^{-1} t + C = -\frac{x}{\sqrt{x^2+8}} + \sinh^{-1}\left(\frac{x}{\sqrt{8}}\right) + C$$

C. Integration of rational functions.

(C.1) Partial fractions

$$\frac{1}{x^3+1} = \frac{1}{3}\left[\frac{1}{x+1} - \frac{x-2}{x^2-x+1}\right]; \qquad \frac{1}{x^3-1} = \frac{1}{3}\left[\frac{1}{x-1} - \frac{x+2}{x^2+x+1}\right].$$

$$\frac{1}{x^4-1} = \frac{1}{2}\left[\frac{1}{x^2-1} - \frac{1}{x^2+1}\right]; \qquad \frac{1}{x^4+1} = \frac{\sqrt{2}}{4}\left[\frac{x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right]^{\star}.$$

[* Use $x^4+1 = (x^2+1)^2 - (\sqrt{2}\,x)^2 = (x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$]

(a) $$\int \frac{1}{x^3+8}\,dx = \frac{1}{4}\int \frac{1}{t^3+1}\,dx \qquad \boxed{\text{(A.2) } x = 2t,\ \tfrac{dx}{dt} = 2}$$

$$= \frac{1}{12}\int \left(\frac{1}{t+1} - \frac{t-2}{t^2-t+1}\right)dt \qquad \boxed{\text{(C.1)}}$$

$$= \frac{1}{12}\ln|t+1| - \frac{1}{12}\left[\frac{1}{2}\ln|t^2-t+1| - \frac{3}{2}\int \frac{1}{t^2-t+1}\,dt\right] \qquad \boxed{\text{(C.2)}}$$

$$= \frac{1}{12}\ln|t+1| - \frac{1}{24}\ln|t^2-t+1| + \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C \qquad \boxed{\text{(C.3)}}$$

$$= \frac{1}{12}\ln\left|\frac{x}{2}+1\right| - \frac{1}{24}\ln\left|\frac{x^2}{4} - \frac{x}{2} + 1\right| + \frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + C$$

(C.2) $$\int \frac{x+a}{x^2+bx+c}\,dx = \left(\tfrac{1}{2}\right)\ln|x^2+bx+c|^{\,*} + \left(a - \tfrac{b}{2}\right)\int \frac{1}{x^2+bx+c}\,dx$$

[* Use $\frac{x+a}{x^2+bx+c} = \left(\frac{1}{2}\right)\frac{2x+b}{x^2+bx+c} + \left(a-\frac{b}{2}\right)\frac{1}{x^2+bx+c}$ and (A.1) with $t = x^2+bx+c,\ \frac{dt}{dx} = 2x+b$]

(C.3) $$\int \frac{1}{x^2+bx+c}\,dx = \begin{cases} \frac{2}{|b^2-4c|^{\frac{1}{2}}}\tan^{-1}\left(\frac{2x+b}{|b^2-4c|^{\frac{1}{2}}}\right)^{**} & \text{if } b^2-4c<0; \\ -\frac{2}{|b^2-4c|^{\frac{1}{2}}}\tanh^{-1}\left(\frac{2x+b}{|b^2-4c|^{\frac{1}{2}}}\right)^{**} & \text{if } b^2-4c>0. \end{cases}$$

[** Use $\frac{1}{x^2+bx+c} = \frac{4}{(2x+b)^2-(b^2-4c)}$ and (A.2) with $|b^2-4c|^{\frac{1}{2}}t = 2x+b,\ \frac{dx}{dt} = \frac{|b^2-4c|^{\frac{1}{2}}}{2}$]

(a) $$\int \frac{1}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C \qquad \boxed{\text{(C.3)}}$$

(b) $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{x}{(x^2+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{1}{2}\int \frac{1}{t^2+\frac{3}{4}}\,dx$ $\boxed{\text{(A.1) } t = x^2+\tfrac{1}{2},\ \tfrac{dt}{dx} = 2x}$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right) + C \qquad \boxed{\text{(C.3)}}$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$$

(C.4) $\displaystyle\int \frac{1}{(1+x^2)^{n+1}}\,dx = \frac{x}{2n(1+x^2)^n} + \frac{2n-1}{2n}\int \frac{1}{(1+x^2)^n}\,dx^{***}$

$$\int \frac{1}{(1-x^2)^{n+1}}\,dx = \frac{x}{2n(1-x^2)^n} + \frac{2n-1}{2n}\int \frac{1}{(1-x^2)^n}\,dx$$

$$\left[^{***}\text{ Use } \frac{d}{dx}\left(\frac{x}{(1+x^2)^n}\right) = -\frac{2n-1}{(1+x^2)^n} + \frac{2n}{(1+x^2)^{n+1}}\right]$$

(a) $\displaystyle\int \sec^5 x\,dx = \int \frac{1}{\cos^5 x}\,dx = \int \frac{\cos x}{(1-\sin^2 x)^3}\,dx = \int \frac{1}{(1-t^2)^3}\,dt$ $\boxed{\text{(A.1) } t = \sin x,\ \tfrac{dt}{dx} = \cos x}$

$$= \frac{t}{4(1-t^2)^2} + \frac{3}{4}\int \frac{1}{(1-t^2)^2}\,dt = \frac{t}{4(1-t^2)^2} + \frac{3}{4}\left[\frac{t}{2(1-t^2)} + \frac{1}{2}\int \frac{1}{(1-t^2)}\,dt\right] \qquad \boxed{\text{(C.4) twice}}$$

$$= \frac{t}{4(1-t^2)^2} + \frac{3t}{8(1-t^2)} + \frac{3}{8}\tanh^{-1} t + C = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \frac{3}{8}\tanh^{-1}\sin x + C$$

(b) $\displaystyle\int \frac{1}{(x^2+4)^3}\,dx = \frac{1}{32}\int \frac{1}{(t^2+1)^3}\,dt$ $\boxed{\text{(A.2) } x = 2t,\ \tfrac{dx}{dt} = 2}$

$$= \frac{1}{32}\left[\frac{t}{4(t^2+1)^2} + \frac{3}{4}\int \frac{1}{(t^2+1)^2}\,dt\right] = \frac{1}{32}\left[\frac{t}{4(t^2+1)^2} + \frac{3}{4}\left[\frac{t}{2(t^2+1)} + \frac{1}{2}\int \frac{1}{t^2+1}\,dt\right]\right] \qquad \boxed{\text{(C.4) twice}}$$

$$= \frac{1}{32}\left[\frac{t}{4(t^2+1)^2} + \frac{3t}{8(t^2+1)} + \frac{3}{8}\tan^{-1} t\right] + C = \frac{x}{16(x^2+4)^2} + \frac{3x}{128(x^2+4)} + \frac{3}{256}\tan^{-1}\frac{x}{2} + C$$