1(10pts). Let \( \mathbf{a} = <1, 2, 3> \), \( \mathbf{b} = <1, 1, 1> \). Find
(a) \( 3\mathbf{a} \cdot \mathbf{b} \), \( \mathbf{a} \times 4\mathbf{b} \);
(b) a vector \( \overrightarrow{BC} \) so that a triangle \( \Delta ABC \) with vertices \( A \), \( B \) and \( C \) has \( \overrightarrow{AB} = \mathbf{a} \) and \( \overrightarrow{AC} = \mathbf{b} \).

Solution: (a) \( 3\mathbf{a} \cdot \mathbf{b} = <3, 6, 9> \cdot <1, 1, 1> = 18; \)
\( \mathbf{a} \times 4\mathbf{b} = <1, 2, 3> \times <4, 4, 4> \)
\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
1 & 2 & 3 \\
4 & 4 & 4 \\
\end{vmatrix} = \begin{vmatrix}
i & 1 & 3 \\
2 & 4 & 4 \\
3 & 4 & 4 \\
\end{vmatrix} + \begin{vmatrix}
\mathbf{i} & 1 & 2 \\
4 & 4 & 4 \\
\mathbf{k} & 4 & 4 \\
\end{vmatrix} = -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k} = < -4, 8, -4 > .
\]
(b) Because \( \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0 \),
\( \mathbf{a} - \mathbf{b} + \overrightarrow{BC} = 0, \overrightarrow{BC} = \mathbf{b} - \mathbf{a} = <1, 1, 1> - <1, 2, 3> = <0, -1, -2> .
\]

2(20pts). Determine if the following two straight lines
\[ x = 1 + 2t, \quad y = 2t, \quad z = 3 - 3t; \quad x = 1 + s, \quad y = s, \quad z = 1 - 3s \]
are skew lines.

Solution: We need to check if they can be on the same plane. Two lines can’t be on the same plane if they are not parallel and they don’t intersect. The direction vectors of the two straight lines are \( \mathbf{v}_1 = <2, 2, -3> \) and \( \mathbf{v}_2 = <1, 1, -3> \). They are not proportional to each other therefore not parallel. To see if they intersect, equating the \( x \), \( y \) and \( z \) of the two straight lines:
\[ 1 + 2t = 1 + s, \quad 2t = s, \quad 3 - 3t = 1 - 3s \]
and solve for \( t \) and \( s \) to get \( t = -\frac{2}{3} \) and \( s = -\frac{4}{3} \). Therefore they intersect at \( x = -\frac{1}{3}, y = -\frac{4}{3} \) and \( z = 5 \) and so they are not skew lines.

Note since there are 3 equations and only two variables, in general there could be no solution.

3(20pts). Find the equation of the plane that passes through \( A(1, 0, 3) \) and the straight line
\[ x = 1 + 2t, \quad y = 3 + t, \quad z = 2 + 2t; \]

Solution: We have a point on the plane and we will find a normal vector of the plane to set up the equation of the plane. We know that the direction vector \( <2, 1, 2> \) of the line
is on the plane. Any point on the line is on the plane. So \( B(1, 3, 2) \) is on the plane and so\( \overrightarrow{AB} = <0, 3, -1> \). A normal vector \( \vec{n} = <2, 1, 1> \times <0, 3, -1> = <-7, 2, 6> \). The equation of the plane is

\[-7(x - 1) + 2y + 6(z - 3) = 0 \quad \text{or} \quad -7 + 2y + 6z = 11\]

4(20pts). For the two planes \( 2x + y - z = 1 \) and \( x - y + z = 3 \),
(a) find the acute angle between them;
(b) find a parametric equation of the straight line of their intersection, if any.

**Solution:** The cosine of the angle between the two normal vectors is

\[\cos \theta = \frac{<2, 1, -1> \cdot <1, -1, 1>}{||<2, 1, -1>|| ||<1, -1, 1>||} = 0\]

therefore \( \theta = \pi/2 \).
(b) The intersection has direction vector

\[\begin{vmatrix}
    \mathbf{i} & \mathbf{j} & \mathbf{k} \\
    2 & 1 & -1 \\
    1 & -1 & 1
\end{vmatrix} = |1 -1 1| \mathbf{i} - |2 -1 1| \mathbf{j} + |2 1 -1| \mathbf{k} = 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k} = <0, -3, -3>\]

To find a point on the line, let \( y = 0 \), then \( x = \frac{4}{3} \) and \( z = \frac{5}{3} \). Therefore the parametric equation for the line is

\[x = \frac{4}{3}, \quad y = -3t, \quad z = \frac{5}{3} - 3t\]

5(10pts). For the given quadratic equation \( x^2 - y^2 + 2x - 4y + z = 0 \),
(a) find its standard form;
(b) write the equation in cylindrical coordinate and in spherical coordinate.

**Solution:** (a) By completing square

\[(x + 1)^2 - (y + 2)^2 = -(z + 3),\]

and letting \( X = x + 1, \ Y = y + 2 \) and \( Z = z + 3 \), we obtain the standard form \( X^2 - Y^2 = -Z \).

(b) Cylindrical coordinate: \( x = r \cos \theta, \ y = r \sin \theta, \ z = z, \)

\[r^2 \cos^2 \theta - r^2 \sin^2 \theta + 2r \cos \theta - 4r \sin \theta + z = 0\]

Spherical coordinate: \( x = \rho \sin \phi \cos \theta, \ y = \rho \sin \phi \sin \theta, \ z = \rho \cos \phi\)

\[\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta + 2\rho \sin \phi \cos \theta - 4\rho \sin \phi \sin \theta + \rho \cos \theta = 0.\]

6(20pts). For the curve \( \mathbf{r} = <2 \sin t, 2 \cos t, t> \),
(a) find the arc length of the curve from \( t = 0 \) to \( t = \frac{\pi}{2} \);
(b) find the curvature \( \kappa \) at \( t = \frac{\pi}{2} \).

**Solution:** \( \mathbf{r}'(t) = <2 \cos t, -2 \sin t, 1> \) and \( \mathbf{r}''(t) = <-2 \sin t, -2 \cos t, 0> \)
(a) The arc length of the curve from $t = 0$ to $t = \frac{\pi}{2}$ is:

$$L = \int_{0}^{\frac{\pi}{2}} |\mathbf{r}'(t)| \, dt = \int_{0}^{\frac{\pi}{2}} \sqrt{4 \cos^2 t + 4 \sin^2 t + 1} \, dt = \frac{\pi \sqrt{5}}{2}$$

(b) The curvature $\kappa$ at $t = \frac{\pi}{2}$ is:

$$\kappa\left(\frac{\pi}{2}\right) = \frac{|\mathbf{r}'\left(\frac{\pi}{2}\right) \times \mathbf{r}''\left(\frac{\pi}{2}\right)|}{|\mathbf{r}'\left(\frac{\pi}{2}\right)|^3} = \frac{|<2 \cos t, -2 \sin t, 1 > \times < -2 \sin t, -2 \cos t, 0 >|}{|<2 \cos t, -2 \sin t, 1 >|^3}$$

$$= \frac{|<2 \cos t, -2 \sin t, 4>|}{5^{3/2}} = \frac{2}{5}$$

Note: $|<2 \cos t, -2 \sin t, 1 > \times < -2 \sin t, -2 \cos t, 0 >|$

$$= \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 \cos t & -2 \sin t & 1 \\
-2 \sin t & -2 \cos t & 0
\end{vmatrix} = 2 \cos t \mathbf{i} - 2 \sin t \mathbf{j} - 4 \mathbf{k} = <2 \cos t, -2 \sin t, -4>.$$