

Vector Analysis—Math 471/571

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Jeffrey Norden & Matthias Eller

§ 1 REVIEW OF VECTORS (and a gentle introduction to linear algebra)

It is certainly correct to say that a vector has both magnitude and direction. It is also correct to say that we can picture vectors as arrows drawn between a pair of points in space. In the greatest generality, the correct way to define a vector is to say that it is a member of a “vector space,” and to then define exactly what a vector space is. For this course, however, we can adopt the following fairly simple definition for what a vector is.

1.1 Definition. Let n be a positive integer. An n -dimensional vector is an ordered list of n real numbers.

We usually use angle bracket notation to denote a specific vector. For example $\langle 2, 7 \rangle$ is a two-dimensional vector (but more precisely, the vector itself is simply “2 followed by 7”), $\langle 3, -1, 8.2 \rangle$ is a three-dimensional vector, and $\langle 1, -2, 3.9, 42, \sqrt{7} \rangle$ is a five-dimensional vector. There are two common notation for indicating that a variable represents a vector. One can use bold face as our text does: $\mathbf{v} = \langle 1, 2, 3 \rangle$, or one can use an arrow as we will do in class and in these notes: $\vec{v} = \langle 1, 2, 3 \rangle$.

We *visualize* a vector as an arrow between two points in the appropriate space (this is a bit of a stretch if the dimension is four or more). So we can picture $\langle 2, 7 \rangle$ as an arrow from the origin which goes right two units and up seven. But we can also picture it starting at another point—the arrow between the points $(1, 4)$ and $(3, 11)$ would also be labeled by $\langle 2, 7 \rangle$. The direction of the vector is just the direction of the corresponding arrow. The magnitude is computed via the Pythagorean Theorem by taking the square root of the sum of the squares, i.e., $\|\langle 2, 7 \rangle\| = \sqrt{2^2 + 7^2} = \sqrt{53}$, while $\|\langle 1, -2, 3.9, 42, \sqrt{7} \rangle\| = \sqrt{1791.21}$. Many authors, including our textbook, just use simple absolute value signs to denote vector length.

You should already be quite familiar with several operations on vectors. If \vec{v} and \vec{w} are vectors of the same dimension, then $\vec{v} + \vec{w}$ is another vector of this same dimension formed by adding coordinate-wise. If t is a number and \vec{v} is a vector, then $t\vec{v}$ is a vector formed by multiplying each coordinate of \vec{v} by t . In this context we usually refer to the number t as a *scalar*, which is just a fancy way of saying that it is a quantity which is not a vector.

So $5\langle 2, 3 \rangle = \langle 10, 15 \rangle$ and $\langle 1, 2, 3 \rangle + \langle 4, 5, 6 \rangle = \langle 5, 7, 9 \rangle$. But note that $\langle 1, 2, 3 \rangle + \langle 4, 5 \rangle$ is undefined since the dimensions are different. These two operations (*vector addition* and *scalar multiplication*) are the most fundamental. You should also be familiar with two vector products. If \vec{v} and \vec{w} are vectors of the same dimension, then we form the *dot product* $\vec{v} \cdot \vec{w}$ by summing the result of coordinate-wise multiplication. This is sometimes also called the scalar product of \vec{v} and \vec{w} , since the result is a scalar. If \vec{v} and \vec{w} are three dimensional vectors, then you should also be familiar with the vector product $\vec{v} \times \vec{w}$. It is important to note that this cross product is defined only in dimension three.

There is another notational device for dealing with vectors. If we let \vec{i} , \vec{j} , and \vec{k} denote the unit coordinate vectors, then we can uniquely decompose any vector into a sum of scalar multiples of these vectors. For example, $\langle 1, 2, 3 \rangle = 1\langle 1, 0, 0 \rangle + 2\langle 0, 1, 0 \rangle + 3\langle 0, 0, 1 \rangle = \vec{i} + 2\vec{j} + 3\vec{k}$. However, one has to be a bit careful with this notation, since it can sometimes make the dimension ambiguous. We write $\langle 2, 3 \rangle = 2\vec{i} + 3\vec{j}$, but $\langle 1, 0 \rangle$ and $\langle 1, 0, 0 \rangle$ are *not* the same vectors. You must tell from the context which of the unit vectors \vec{i} and \vec{j} actually represent. So, when you see $2\vec{i} + 3\vec{j}$ you must be able to decide whether this means the vector $\langle 2, 3 \rangle$ or the vector $\langle 2, 3, 0 \rangle$. (Fortunately, this sort of notation is never used in dimensions greater than three.)

You may have noticed that something about our definition of vector seems very familiar or even a bit suspicious. Recall that \mathbb{R}^2 denotes the set of all points in the plane, and \mathbb{R}^3 denotes the set of all points in three dimensional space. Well, the points in \mathbb{R}^2 were *also* defined to be ordered pairs of real numbers, and points in \mathbb{R}^3 were defined as ordered triples of real numbers. How can we use the same definition for for points and vectors? It *seems* like vectors and points are very different. In fact the opposite is actually true:

1.2 Note. *Vectors and points are exactly the same mathematical objects!*

In other words, there is only one “ordered pair consisting of the number 2 followed by the number 3.” This isn’t changed by the fact that we now have two different notations for it: $(2, 3)$ and $\langle 2, 3 \rangle$. How can an ordered pair simultaneously represent both a point and arrow? This sort of thing actually happens all the time. The number “2” might represent 2 apples or 2 oranges, we don’t use a different number system for each type of thing we want to count. For a better example, consider a simple function like $f(x) = x^2$. When we write $f(\sqrt{2}) = 2$, the 2 on the right hand side represents the idea that “ f has a magnitude of 2” at a certain x -value. When we write $f'(1) = 2$, this exact same

“number 2” represents the rate at which f is changing at a certain x -value. In the same way, the ordered pair “2 followed by 3” can represent either a point or the idea of moving 2 units to the right and then 3 units up.

Actually we could make do with just one notation, and many mathematicians use just $\langle 2, 3 \rangle$, leaving it to the context in which it is used to determine whether it should be a point or a vector. More important is that we use a different notation for variables: when you see either $\mathbf{x} = \mathbf{y} + \mathbf{z}$ (in the textbook) or $\vec{x} = \vec{y} + \vec{z}$ (in these notes) it *means* that in addition to whatever else is said, \vec{x} , \vec{y} , and \vec{z} are vectors. This saves a lot of tedious writing of phrases like “let x , y , and z be vectors.”

Why is this important to understand? The fact that we can interchange vectors and points is the key to using vectors effectively. Consider the derivative example again. Even though each $f'(x)$ represents a rate, we still draw the graph of f' in order to understand its properties. We can do this because, regardless of how we interpret them, the values of $f'(x)$ are numbers. Suppose we are working in \mathbb{R}^2 , and consider solving the vector equation $\|\vec{x}\| = 1$. More precisely, consider the set defined by

$$S = \{ \vec{x} : \vec{x} \in \mathbb{R}^2 \text{ and } \|\vec{x}\| = 1 \}$$

Then S is a set of vectors, but how should we interpret and picture this set? We will do so by considering S to be a set of points, so $\|\vec{x}\| = 1$ becomes an equation for the unit circle in \mathbb{R}^2 . This is just the boundary of the circle—to include the inside, you would use $D = \{ \vec{x} : \vec{x} \in \mathbb{R}^2 \text{ and } \|\vec{x}\| \leq 1 \}$. Note that if we change \mathbb{R}^2 to \mathbb{R}^3 , the same equation defines the unit sphere in three dimensions. In even higher dimensions we take this equation to be the definition of the unit sphere, even though it is virtually impossible to picture it (such things are sometimes called hyper-spheres). Even more important are vector definitions for lines and planes such as $L = \{ t \langle 2, 4 \rangle : t \in \mathbb{R} \}$ or $P = \{ \vec{x} : \vec{x} \cdot \langle 1, 5, 7 \rangle = 0 \}$.

There is another way to view the distinction between points and vectors. The field of mathematics called linear algebra studies the general notion of a vector space. A vector space is a set V together with two operations defined on it: scalar multiplication and vector addition. So for any number t and any \vec{v} in V there must be a well defined $t\vec{v}$ in V , and for any \vec{v} and \vec{w} in V , there must be a well defined $\vec{v} + \vec{w}$ in V . The set V and the multiplication and addition defined on it can be quite general and abstract. But, in order to earn the title of “vector space” they are required to satisfy certain algebraic

properties, similar to the commutative and distributive properties that the real numbers have. (If you have your copy of Stewart's calculus textbook handy, these properties are exactly the ones which are listed in the box at the bottom of page 692).

For an example of an abstract vector space, consider letting V denote the set of all continuous functions from \mathbb{R} to \mathbb{R} , and define tf and $f + g$ by $(tf)(x) \stackrel{\text{def}}{=} tf(x)$ and $(f + g)(x) \stackrel{\text{def}}{=} f(x) + g(x)$. This does satisfy the required algebraic properties (check this), but it is not a 2-dimensional, or even a 200-dimensional space. In an important sense, this space is *infinite* dimensional.

We won't be able to go very far into the general theory of linear algebra this semester, but you are likely to encounter it in later classes. For now, the important idea is very simple. When we think of \mathbb{R}^2 as the plane with its geometric properties (such as distance) we call it "Euclidean space" and refer to its elements as "points." But if we think of \mathbb{R}^2 as a set on which scalar multiplication and vector addition are defined, we call it a "vector space" and refer to its elements as "vectors."

1.3 Review Problems.

Decide which of the following are well defined, and evaluate those that are. Give a geometric interpretation for each of the defined answers.

- 1) $2\langle 1, -2, \pi \rangle - \sqrt{2}\langle 8, 4, -7 \rangle$
- 2) $\langle 1, 3 \rangle + \langle 5, 2, 1 \rangle$
- 3) $\langle 1, 4 \rangle \langle 6, 2 \rangle$
- 4) $\langle 1, 4 \rangle \cdot \langle 6, 2 \rangle$
- 5) $\langle 1, 4 \rangle \cdot \langle 6, 2, 3 \rangle$
- 6) $\langle 1, 4 \rangle \times \langle 6, 2 \rangle$
- 7) $\langle 1, 4, 1 \rangle \times \langle 6, 2, 3 \rangle$
- 8) $\langle 1, 4, 1 \rangle \cdot \vec{i}$
- 9) $(2\vec{i} + 3\vec{j}) \cdot (7\vec{i} - 8\vec{j})$
- 10) $(2\vec{i} + 3\vec{j}) \times (\vec{i} - \vec{j} + 2\vec{k})$
- 11) $\langle 1, 2, 3, 4, 5, 6, 7 \rangle \cdot \langle 7, 6, 5, 4, 3, 2, 1 \rangle$
- 12) $\| \langle 1, 2, 3, 4, 5, 6, 7 \rangle \|$
- 13) $\| \langle \sqrt{2}, 10, 1/\pi \rangle \|$
- 14) $\vec{r} \cdot \vec{r} - \|\vec{r}\|^2$