TOPOLOGY—Math 431/531
FALL 1999

Tennessee Technological University   Jeffrey Norden

§ **0** *INTRODUCTION*

Our class meets TR from 12:00-1:20 in room BR 420. My office is room BR 107. My "official" office hours are Monday and Friday from 1–2pm and Wednesday from 2-3pm, but I am available at other times as well. To contact me outside of class you can call my office at x3592 or send email to me at `jnorden@tntech.edu`.

This class will be taught in a style which is probably different from other mathematics courses you have had. There will be no textbook. You will receive notes containing definitions, statements, and some supplementary remarks. The statements are divided into two categories: labeled either "theorems" or "propositions." Your assignment is to provide proofs of the theorems (which are true unless I have made a typographic error) and either proofs or disproofs of the propositions (some of which are true and others of which are false).

This should not be done by looking up proofs in textbooks or papers. You will instead use the mathematical ability and knowledge you have learned in other classes and the help provided by the instructor and your fellow classmates both in and out of class. The majority of class time will be spent providing this help. You will also be asked to write up some of your proofs and distribute them to the class.

The motivation for teaching a class in this way is that you will learn to *do* mathematics instead of just learning about mathematics. The position you are placed in is similar to that of a research mathematician—you have certain facts to draw on and there are certain facts which you are trying to prove. Experience has shown that topology is a subject which is especially well suited to this method of teaching, perhaps because so many of the proofs have a "natural" flow to them. There is a long history of teaching topology in this way, starting with R. L. Moore in the 1920's. The disadvantage of this method is that we cannot cover as much material as a textbook based course can. At some point, we may address this problem by reading some book excerpts and/or research papers.

Assigning a grade can sometimes be difficult in a course like this. Ideally, everyone works hard enough to earn an A (this would certainly make my job easier!). We will have a midterm and final which together will count for about 1/4 of your final grade. But, the bulk of your grade will depend upon the work you do in trying to prove and disprove statements. Since I don't like to discuss individual grades in front of the class, it is up to you to talk with me outside of class inquire about whether your progress is satisfactory. You should feel free to do this at any time.

---

§ **1** *STARTING OUT*

In order to start quickly, for this section we will make use of what you already know about the "spaces" $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. So for this section, the terms "point" and "set" will refer only to elements or subsets of one of these three spaces.

1.1 Definition. Let A be a set and let p be a point. Then *p is a limit point of A* means that for every positive number ε there is a point a of A such that the distance from p to a is less than ε and $p \neq a$ (Some authors use the terms "cluster point" or "accumulation point" instead of limit point.)

The proof of the following theorem gives a good introduction to what proofs in topology are like.

Theorem 1. *Let A be a set. Then every limit point of the set of limit points of A is a limit point of A.*

Proof. Fix a set A, and let L denote the set of points which are limit points of A. Now fix a point x which is a limit point of L. We need to show that x is a limit point of A. Fix a positive number ε. We will be done if we can find a point of A which is not equal to x and is closer than ε to x.

Since x is a limit point of L, we can choose a point y of L such that $d(x,y) < \varepsilon$ and $y \neq x$, where $d(x,y)$ denotes the distance from x to y. Note that $d(x,y) > 0$ and $d(x,y) < \varepsilon$, so we can choose a positive number ε' such that $\varepsilon' < \min\{\, d(x,y), \varepsilon - d(x,y)\,\}$. Since y is a point of L, y is a limit point of A, so we can choose a point a of A such that $d(y,a) < \varepsilon'$. We claim that a is the point of A we are looking for. By the triangle inequality (see Theorem 2, below) we have that

$$d(x,a) \leq d(x,y) + d(y,a) < d(x,y) + \varepsilon' < d(x,y) + \varepsilon - d(x,y) = \varepsilon$$

so the distance from x to a is less than ε. Finally, $d(x,a) < \varepsilon' < d(x,y)$, so $d(x,a) \neq d(x,y)$, and thus $a \neq y$. □

Note: drawing a *picture* to illustrate the arguments used above is absolutely essential for understanding the ideas which lead to the proof.

1.2 Definition. We will always use $d(x,y)$ to denote the distance between the points x and y. To be precise:

$$\text{For } \mathbb{R} := d(x,y) \stackrel{\text{def}}{=} |x-y|$$
$$\text{For } \mathbb{R}^2 = d(x,y) \stackrel{\text{def}}{=} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$
$$\text{For } \mathbb{R}^3 = d(x,y) \stackrel{\text{def}}{=} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$$

where x_1, x_2, and x_3 denote the first, second, and (possibly) third coordinates of the point x.

We have already used the fact that $d(x,y)$ satisfies the triangle inequality, but this really does require proof.

Theorem 2. *For any points x, y, and z, $d(x,z) \leq d(x,y) + d(y,z)$.*

1.3 Notations and Definitions. The notations $A \subset B$ and $A \subseteq B$ both denote that A is a subset of B, i.e., every point which is in A is also in B (the second notation merely emphasizes the fact that every set is considered to be a subset of itself). The notations $A \cap B$ and $A \cup B$ indicate the intersection and union of A and B. More generally, if A_1, A_2, A_3, $\ldots$ is an infinite sequence of sets, then we define the infinite intersection and union of the sequence:

$$\bigcap_{n=1}^{\infty} A_i \stackrel{\text{def}}{=} \{\, x : s \in A_i \text{ for every } i = 1,2,3,\ldots \}$$
$$\bigcap_{n=1}^{\infty} A_i \stackrel{\text{def}}{=} \{\, x : s \in A_i \text{ for every } i = 1,2,3,\ldots \}$$

In general, we will use $L(A)$ to denote the set of *all* limit points of the set A (this is sometimes called the first derived set of A). Note that it may happen that A has no limit points, in which case $L(A) = \varnothing$, where $\varnothing$ denotes the "empty set" which has no elements. We let $L^2(A) = L(L(A))$, and for any integer $n > 2$, let $L^n(A) = L(L^{n-1}(A))$ (also known as the n^{th} derives set of A). As a convenience, we also let $L^1(A) = L(A)$ and $L^0(A) = A$. Finally, we let $L^\infty(A)$ denote $\bigcap_{n=1}^{\infty} L^n(A)$.

Note that Theorem 1 can now be expressed simply as $L^2(A) \subset L(A)$. This can be generalized by applying the technique of mathematical induction.

Theorem 3. *For every integer $n \geq 1$, $L^{n+1}(A) \subset L^n(A)$.*

Theorem 4. *For every integer integer* $n \geq 1$ *there is a set* A *such that* $L^n(A) \neq \varnothing$ *and* $L^{n+1}(A) = \varnothing$

Theorem 5. *There is a set* A *such that* $L^n(A) \neq \varnothing$ *for every positive integer* n *and* $L^\infty(A) = \varnothing$.

1.4 Notations and Definitions. If x is a point and $\varepsilon > 0$, the *open ball centered at* x *with radius* ε is denoted by $B_\varepsilon(x)$ and defined as $B_\varepsilon(x) \stackrel{\text{def}}{=} \{\, p : p \text{ is a point and } d(x,p) < \varepsilon \,\}$. The *complement* of a set A is the set of all points which are not in A; we will denote it by A'. A set A is *closed* means that every limit point of A is also a point of A, i.e., $L(A) \subset A$. A set A is *open* means that for every point x in A there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.

Theorem 6. *A set is open iff its complement is closed. A set is closed iff its complement is open.*

Theorem 7. *Every open ball is an open set. Every closed ball is a closed set, where a closed ball is defined to be a set of the form* $\{\, p : p \text{ is a point and } d(x,p) \leq \varepsilon \,\}$.

Theorem 8. *Let* A *and* B *be closed sets. Then* $A \cap B$ *and* $A \cup B$ *are closed sets.*

Theorem 9. *Let* A *and* B *be open sets. Then* $A \cap B$ *and* $A \cup B$ *are open sets.*

1.5 Notations for some sets. We will adopt the following notation for this class (most of which are quite standard). The notation $x \in A$ means that the point x is a member, or element, of the set A. The symbol $\mathbb{Z}$ denotes the set of all integers ($\mathbb{Z} = \{\, 0, 1, -1, 2, -2, \ldots \,\}$), $\mathbb{N}$ denotes the strictly positive integers ($\mathbb{N} = \{\, 1, 2, 3, \ldots \,\}$) and ω denotes the the non-negative integers ($\omega = \mathbb{N} \cup \{0\}$). Also, $\mathbb{R}$ denotes the set of all real numbers (we will say more later about what the term "real number" really means), $\mathbb{Q}$ denotes the set of all rational numbers ($\mathbb{Q} = \{\, x \in \mathbb{R} : x = n/m \text{ for some } n, m \in \mathbb{Z} \,\}$), and $\mathbb{P}$ denotes the set of all irrational numbers ($\mathbb{P} = \{\, x \in \mathbb{R} : x \notin \mathbb{Q} \,\}$).

1.6 Definition. Let A_1, A_2, A_3, ... be an infinite sequence of sets. Then the infinite union and the infinite intersection of this sequence are defined by:

$$\bigcup_{n=1}^{\infty} A_n \stackrel{\text{def}}{=} \{\, x : x \in A_n \text{ for some } n \in \mathbb{N} \,\} \qquad \bigcap_{n=1}^{\infty} A_n \stackrel{\text{def}}{=} \{\, x : x \in A_n \text{ for every } n \in \mathbb{N} \,\}$$

1.7 Definition. A set A is *bounded* means that there is some point x and some number $r > 0$ such that $A \subset B_r(x)$.

Proposition 10. *Let A_1, A_2, ... be an infinite sequence of closed sets. Then: (a) $\bigcup_{n=1}^{\infty} A_n$ is a closed set, and (b) $\bigcap_{n=1}^{\infty} A_n$ is a closed set.*

Proposition 11. *Let A_1, A_2, ... be an infinite sequence of open sets. Then: (a) $\bigcup_{n=1}^{\infty} A_n$ is an open set, and (b) $\bigcap_{n=1}^{\infty} A_n$ is an open set.*

Proposition 12. *Let A_1, A_2, ... be an infinite sequence of non-empty closed and bounded sets. Then $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$. Furthermore, the condition that the sets are bounded is necessary.*

Proposition 13 (Bruggink's proposition). *Let A and B be sets with $A \subset B$. Then $L(A) \subset L(B)$.*

§ 2 *METRIC SPACES*

2.1 Definition. A *metric space* is an ordered pair $\langle X, d\rangle$ where X is a set and d is a function which assigns a non-negative real number to each pair of points in X (i.e., $d : X \times X \to [0, \infty)$) such that for all x, y, and z in X:

(i) $d(x, y) = d(y, x)$
(ii) $d(x, y) = 0$ if, and only if, $x = y$,
(iii) $d(x, z) \leq d(x, y) + d(y, z)$

Condition 3 is called the triangle inequality. The function d is called a metric or distance function for X (when it satisfies all the conditions). When the distance function is understood or unimportant will abuse notation and refer to the metric space X instead of the metric space $\langle X, d\rangle$ (i.e., the metric space $\mathbb{R}$ will mean $\langle \mathbb{R}, d\rangle$ where $d(x, y) = |x - y|$). We will refer to the elements of X as the points of the space.

Since the definitions and theorems from section 1 only referred to distance, they all generalize to the setting of metric spaces. From now on, the terms "point" and "set" will refer to points and sets in metric spaces, unless we state otherwise. To avoid unnecessary repetition, we will just add an "m" to each of the definitions and theorems from section 1 in order to refer to the corresponding more general statement. So, Definition 1.1m defines a limit point of a set in a metric space, and similarly 1.3m, 1.4m, and 1.7m define $L(A)$,

$B_\varepsilon(x)$, closed, open, and bounded for metric spaces. Theorems 1m, 3m, 6m–9m, and Propositions 10m–13m are similarly the original statements generalized to metric spaces. Theorem 2 needs to be interpreted slightly differently—it says that the usual distances for $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ make those spaces into valid metric spaces.

Proposition 14. *Each of the following is a metric for* $\mathbb{R}$*:*

(a) $d(x,y) = 36.7\,|x-y|$
(b) $d(x,y) = |x+y|$
(c) $d(x,y) = 1$ for all $x, y \in \mathbb{R}$
(d) $d(x,y) = \tan^{-1}(|x-y|)$
(e) $d(x,y) = |\tan^{-1}(x) - \tan^{-1}(y)|$
(f) $d(x,y) = \frac{|x-y|}{1+|x-y|}$

Proposition 15. *There is a metric for* $\mathbb{R}^2$ *such that each* $B_\varepsilon(x)$ *is "square shaped."*

Proposition 16. *(a) There is a metric for* $\mathbb{R}$ *such that* $d(x,y) = |x-y|$ *if* x *and* y *are both rational or if* x *and* y *are both irrational, but for which both* $\mathbb{Q}$ *and* $\mathbb{P}$ *are closed sets. (b) There is a metric for* $\mathbb{R}^2$ *such that* $d(x,y)$ *is the usual distance if the line connecting the points* x *and* y *passes through the origin, but such that* $x \notin L(A)$ *whenever* A *is a set which does not intersect the line through the origin which contains* x*.*

Proposition 17 (Howard's proposition). *If* A *and* B *are sets in a metric space, then* $L(A \cup B) = L(A) \cup L(B)$ *and* $L(A \cap B) = L(A) \cap L(B)$*.*

2.2 Definition. The *closure* of a set A is defined by $\mathrm{cl}(A) \stackrel{\text{def}}{=} A \cup L(A)$.

The closure of a set is the "smallest" closed set which contains A. This is made precise by:

Theorem 18. *For any set* A, $\mathrm{cl}(A)$ *is a closed set containing* A*. If* A *is a set and* B *is any closed set which contains* A*, then* $\mathrm{cl}(A) \subset B$*.*

2.3 Definition. Let A be a subset of a metric space X, and let $x \in X$. Then:

- *x is an interior point of A* means that there exists a positive number ε such that $B_\varepsilon(x) \subset A$.
- *x is a boundary point of A* means that for every positive number ε, $B_\varepsilon(x)$ contains a point which is in A and a point which is not in A.

- *x is an exterior point of A* means that there exists a positive number ε such that $B_\varepsilon(x)$ contains no points of A.

We will use $\mathrm{int}(A)$, $\mathrm{bd}(A)$, and $\mathrm{ext}(A)$ to denote the set of all interior, boundary, and exterior points of a set A (note that any of these might be the empty set).

Theorem 19. *Let A be a subset of a metric space X, and let $x \in X$. Then x is either an interior point of A, a boundary point of A, or an exterior point of A. In fact, x is exactly one of these.*

Note that a set is open iff every one of its points is an interior point. More than this, we have that:

Theorem 20. *The sets* $\mathrm{int}(A)$ *and* $\mathrm{ext}(A)$ *are always open, and the set* $\mathrm{bd}(A)$ *is always closed.*

Theorem 21. *For any set A,* $A \cup \mathrm{bd}(A) = \mathrm{cl}(A)$.

Proposition 22. *There is no general subset or intersection relationship between A and any one of the sets* $\mathrm{int}(A)$, $\mathrm{bd}(A)$, *and* $\mathrm{ext}(A)$.

Proposition 23. *There exists a non-trivial subset A of $\mathbb{R}$ such that* $\mathrm{bd}(A) = \varnothing$. *(By non-trivial, we mean $A \neq \mathbb{R}$ and $A \neq \varnothing$.)*

2.4 Definition. Let $A \subset \mathbb{R}$. A number x is an *upper bound for A* means that $x \geq a$ for all $a \in A$. A number x is a *least upper bound for A* means that x is an upper bound for A and that $x \leq y$ for every upper bound y for A. Similarly, the notions of *lower bound* and *greatest lower bound* are defined by reversing the inequalities above.

Theorem 24. *Least upper bounds and greatest lower bounds are unique (if they exist). More precisely, if x and y are both least upper bounds (or greatest lower bounds) for the set $A \subset \mathbb{R}$, then $x = y$.*

2.5 Notation. *The least upper bound of a set A (if it exists) is also called the supremum of A, and is denoted by either* $\sup(A)$ *or* $\mathrm{lub}(A)$. *Similarly, the greatest lower bound is also called the infinum and is denoted by* $\inf(A)$ *or* $\mathrm{glb}(A)$.

Proving the existence of $\sup(A)$ will require a careful construction of the real number system, which is a somewhat tedious task that we will put off till later. So, for now, we will accept the following "axiom" which says basically that the real number system doesn't have any "holes" in it.

2.6 Axiom. *Every set of real numbers which has an upper bound has a least upper bound.*

However, we only need this one axiom. Once we adopt it, we can then prove the corresponding statement for lower bounds.

Theorem 25. *Every set of real numbers which has a lower bound has a greatest lower bound.*

There are many examples of metric spaces which are more abstract and more difficult to "picture" than the one's we have looked at so far. Here is a start at looking at them.

Proposition 26. *Let F denote the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ (by bounded, we mean simply that the set $\operatorname{ran}(f) \overset{\text{def}}{=} \{\, f(x) : x \in \mathbb{R} \,\}$ has both an upper and a lower bound). Let $d(f, g) = \sup\{\, |f(x) - g(x)| : x \in \mathbb{R} \,\}$. Then $\langle F, d \rangle$ is a metric space.*

Proposition 27. *Let S denote the set of all non-empty bounded subsets of $\mathbb{R}^2$. For any $A \in S$ and any point $p \in \mathbb{R}^2$, let $d(p, A) \overset{\text{def}}{=} \inf\{\, d(p, a) : a \in A \,\}$, where the second d denotes the usual distance function in the plane. Now, for any sets $A, B \in S$, let $d(A, B)$ be the maximum of $\sup\{\, d(a, B) : a \in A \,\}$ and $\sup\{\, d(b, A) : b \in B \,\}$. Then $\langle S, d \rangle$ is a metric space.*

Proposition 27. *Let S denote the set of all non-empty bounded subsets of $\mathbb{R}^2$. For any $A \in S$ and any point $p \in \mathbb{R}^2$, let $d(p, A) \overset{\text{def}}{=} \inf\{\, d(p, a) : a \in A \,\}$, where the second d denotes the usual distance function in the plane. Now, for any sets $A, B \in S$, let $d(A, B)$ be the maximum of $\sup\{\, d(a, B) : a \in A \,\}$ and $\sup\{\, d(b, A) : b \in B \,\}$. Then $\langle S, d \rangle$ is a metric space.*

§ **3** *BACKGROUND ON SETS AND FUNCTIONS.*

A *set* is a collection of "objects". No further *definition* of set is really possible, since "set" is usually considered the basic undefined notion of modern mathematics. Instead of defining what a set is, one can study axiom systems that sets should satisfy, much in the same way that Euclid studied axioms for geometry as opposed to *defining* what points and angles are. For now, we simply need some terms and definitions about sets. This treatment is often referred to as "naive" set theory. We may get into some of the more difficult and interesting properties later in the semester.

The objects which make up a set are called the *elements* of the set. The notation $x \in X$ means that x is an element of the set X. If X and Y are sets, then $X \subset Y$ is read "X is a *subset* of Y", and simply means that every element of X is an element of Y. Note that every set is a subset of itself. If $X \subset Y$ and $X \neq Y$, we write $X \subsetneq Y$, and say that X is a *proper* subset of Y. The notation $X \subseteq Y$ is synonymous with $X \subset Y$, but is sometimes used to emphasize that X might in fact be equal to Y. (In olden days, some authors used $\subset$ for $\subsetneq$ in order to ease the burden on typesetters.)

Curly braces are usually used to indicate sets. The notation $\{ 0, 1, 2, 4 \}$ stands for the set whose elements are exactly the integers 0, 1, 2, and 4. The notation $\{ x : \varphi(x) \}$ stands for the set of all objects which satisfy the "predicate" φ. For example,

$$\{ x : x \text{ is a real number and } x > 0 \text{ and } x < 1 \}$$

defines the open interval $(0, 1)$. (For now, we won't worry about exactly what a "predicate" is; we'll just use common sense.) A useful abbreviation is $\{ x \in A : \varphi(x) \}$ which just stands for $\{ x : x \in A \ \& \ \varphi(x) \}$.

You are already familiar with certain mathematical sets: the set $\mathbb{R}$ of real numbers, the set $\mathbb{Z}$ of integers, the set $\mathbb{Q}$ of rational numbers, and also the empty set $\varnothing$ which is the unique set that has no elements in it. While we won't go into (for now) exactly what sort of objects can and can't be elements of sets, it is important to realize that sets can themselves serve as elements of other sets. For example, define some open intervals in the real line by the definition

$$I_n = (n, n+1) = \{ x \in \mathbb{R} : n < x < n+1 \}$$

where n is an integer. Then we can form sets such as $\{ I_0, I_1, I_2, I_4 \}$ or even $\{ I_n : n = 0, 1, 2, \ldots \}$. The term *collection* is synonymous with set, but it is often used because it is less confusing to think of a "collection of sets" instead of a "set of sets". Of course, once we have collections of sets, we can form collections of collections, etc, so the distinction between set and collection only helps for "one level". As an example, you should convince yourself that $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, etc are all *different* objects.

The unordered pair containing the elements x and y is simply the set $\{x, y\}$. The *ordered pair* $\langle x, y\rangle$ is usually thought of as the elements x and y with x "first" and y "second". This can be given a precise definition as follows:

3.1 Definition.

$$\langle x, y\rangle \stackrel{\text{def}}{=} \{\, \{x\},\ \{x, y\}\, \}$$

In order to see that this definition captures the idea of ordered pair, it is only necessary to prove the following.

Theorem 28. *For any x, y, x', and y', $\langle x, y\rangle = \langle x', y'\rangle$ if, and only if, $x = x'$ and $y = y'$.*

Proposition 29. *If we throw out 3.1 and instead adopt the alternate definition $\langle x, y\rangle = \{\, x,\ \{y\}\, \}$, then Theorem 28 will still hold.*

Another common notation for ordered pair is (x, y). The advantage to using angle brackets instead of parenthesis is that we will avoid confusing pairs and intervals, i.e., $\langle 0, 1\rangle$ is the ordered pair and $(0, 1)$ is the open interval $\{\, x \in \mathbb{R} : 0 < x < 1\,\}$.

A *relation* is a set of ordered pairs. If R is a relation then

$$\operatorname{dom}(R) \stackrel{\text{def}}{=} \{\, x : \text{there is some y such that } \langle x, y\rangle \in R\,\}$$
$$\operatorname{ran}(R) \stackrel{\text{def}}{=} \{\, y : \text{there is some x such that } \langle x, y\rangle \in R\,\}$$

The sets $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are called the *domain* and *range* of the relation R. The relation thus tells us which elements of the domain are "related" to which elements of the range.

The largest possible relation between the sets X and Y is:

3.2 Definition.

$$X \times Y \stackrel{\text{def}}{=} \{\, \langle x, y\rangle : x \in X \ \ \&\ \ y \in Y\,\}$$

This is called the *Cartesian product* of X and Y after René Descartes.

We usually think of a function as a rule or assignment of an element of the range to each element of the domain. This can be made more precise by defining a function to be a relation which relates exactly one element of its range to each element of its domain.

3.3 Definition. A *function* is a relation f such that for each $x \in \operatorname{dom}(f)$ there is a unique $y \in \operatorname{ran}(f)$ such that $\langle x, y\rangle \in f$.

This definition makes a function into a concrete object (a set) instead of an abstract rule. In order to emphasize that we *think* of functions as rules, we use the notation $f(x) = y$ as an abbreviation for the statement $\langle x, y\rangle \in f$. We also use the notation $f : X \to Y$ in order to introduce functions. This is read "f is a function from X into Y", and its meaning is: f is a function, $\mathrm{dom}(f) = X$, and $\mathrm{ran}(f) \subset Y$. Only requiring that $\mathrm{ran}(f)$ be a subset of Y gives us the freedom to write such things as "define $f : \mathbb{R} \to \mathbb{R}$ by the equation $f(x) = x^2$", which really just shorthand for $f = \{ \langle x, y\rangle : x, y \in \mathbb{R} \ \& \ y = x^2 \}$ (note that $\mathrm{ran}(f) = [0, \infty)$). If Y is equal to $\mathrm{ran}(f)$ we say that f is a function from X *onto* Y. Note that this way of looking at a function makes it clear that a function is more than an equation. If we define $g\colon (0, 1) \to \mathbb{R}$ by $g(x) = x^2$, then $g \neq f$ even though the "equations" for f and g are the same. In fact, $g \subsetneq f$! On the other hand, if we define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = (x + 1)^2 - 2x - 1$ then (as sets) $h = f$.

A useful exercise in understanding this definition of function is to look at inverses and compositions.

3.4 Definitions. If R is a relation, then the *inverse* of R is defined by

$$R^{-1} \stackrel{\text{def}}{=} \{ \langle y, x\rangle : \langle x, y\rangle \in R \}.$$

and if R_1 and R_2 are relations, then their *composition* is defined by

$$R_2 \circ R_1 \stackrel{\text{def}}{=} \{ \langle x, z\rangle : \text{there exists some } y \text{ such that } \langle x, y\rangle \in R_1 \text{ and } \langle y, z\rangle \in R_2 \}.$$

(The order is "reversed" to make the definition agree with the standard $f \circ g(x) = f(g(x))$ equation). A function f is 1:1 means that no two members of f can have the same second element and different first elements, i.e., if $x, y \in \mathrm{dom}(f)$ and $f(x) = f(y)$ then $x = y$.

Theorem 30. *Let f be a function. The following statements are equivalent:*

(1) *f is 1:1.*
(2) *f^{-1} is a function.*
(3) *There is a function g such that $f \circ g = \mathrm{id}_{\mathrm{dom}(f)}$ (the function $\mathrm{id}_X \stackrel{\text{def}}{=} \{ \langle x, x\rangle : x \in X \}$ is the identity function on the set X).*

Proposition 31. *Let $f : X \to Y$. The following statements are equivalent:*

(1) *f is a 1:1 function from X onto Y.*
(2) *There is a function g such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.*

Finally, we review the concepts of union, intersection, and set subtraction. If A and B are sets, then

$$A \cap B \stackrel{\text{def}}{=} \{\, x : x \in A \ \ \& \ \ y \in B \,\}$$
$$A \cup B \stackrel{\text{def}}{=} \{\, x : x \in A \text{ or } y \in B \,\}$$
$$A \setminus B = A - B \stackrel{\text{def}}{=} \{\, x : x \in A \ \ \& \ \ y \notin B \,\}$$

Note that the complement of set in a metric space (1.4) is just a special case of subtraction, where the set X is "understood" to be the whole space. We have already introduced the generalizations of intersection and union to an infinite sequence of sets. This generalizes further to the concept of *arbitrary* unions and intersections.

3.5 Definition. If $\mathcal{C}$ is any collection of sets (i.e., a set each of whose elements is itself a set), then

$$\bigcap \mathcal{C} \stackrel{\text{def}}{=} \{\, x : x \in X \text{ for every } X \in \mathcal{C} \,\}$$

$$\bigcup \mathcal{C} \stackrel{\text{def}}{=} \{\, x : x \in X \text{ for some } X \in \mathcal{C} \,\}$$

It is easy to see that this generalizes the definitions in 1.3, since $\bigcap_{i=0}^{\infty} X_i = \bigcap\{\, X_0, X_1, \dots \}$ and $\bigcup_{i=0}^{\infty} X_i = \bigcup\{\, X_0, X_1, \dots \}$. For a trivial example which is not a sequence, let $\mathcal{C}$ denote the collection of all open intervals in $\mathbb{R}$ which contain 0. Then $\bigcap \mathcal{C} = \{0\}$ and $\bigcup \mathcal{C} = \mathbb{R}$.

§ 4 *CONNECTEDNESS.*

4.1 Definition. Let A be a subset of a metric space X. Then A is *disconnected* means that $A = B \cup C$, where B and C are non-empty and neither B nor C contains a point or limit point of the other. In this case we say that the pair $\{B, C\}$ forms a *separation* of A. The set A is *connected* means that it is not disconnected. The space X is a *connected metric space* means that X is a connected subset of itself.

Theorem 32. *$\mathbb{R}$ is a connected metric space.*

Theorem 33. *A subset of $\mathbb{R}$ is connected iff it is an interval, i.e.: (a,b), $[a,b]$, $(a,b]$, $[a,b)$, $[a,\infty)$, (a,∞), $(-\infty,a]$, $(-\infty,a)$, or $(-\infty,\infty)$.*

Theorem 34. *$\mathbb{R}^2$ is a connected metric space.*

4.2 Definition. A subset of a metric space is *clopen* means that it is simultaneously open and closed.

Note that $\varnothing$ is clopen, and that in the metric space X, X itself is a clopen set.

Theorem 35. *A metric space X is connected iff $\varnothing$ and X are the only clopen sets in X.*

Note that a pair $\{B, C\}$ of non-empty sets is a separation of $B \cup C$ iff $B \cap \mathrm{cl}(C) = \mathrm{cl}(B) \cap C = \varnothing$. On the other hand $\{(0,1), (1,2)\}$ is a separation of its union, so we can't simplify this condition to $\mathrm{cl}(B) \cap \mathrm{cl}(C) = \varnothing$. This example also shows that the union of two connected sets may be disconnected.

Proposition 36. *If A and B are connected sets and $A \cap B \neq \varnothing$ then $A \cup B$ is connected and $A \cap B$ is connected.*

Proposition 37. *If $A_1, A_2, A_3, \ldots$ is an infinite sequence of connected sets such that $A_1 \supset A_2 \supset A_3 \supset \ldots$, then $\bigcap_{i=1}^{\infty} A_i$ is connected if it is non-empty.*

Proposition 38. *If A is connected, then $\mathrm{cl}(A)$ is connected.*

§ 5 *RELATIVISATION*

Obviously a given set X can have more than one metric (e.g., Proposition 14). This is one way of creating a "new" metric space out of an old one. But, sometimes the difference between two metrics on a set is not "topological," in that it doesn't effect the limit points, etc.

Theorem 39. *Let X be a set and let d_1 and d_2 be metrics on X. The following conditions are equivalent:*

(1) *For any $A \subset X$ and any $x \in X$, x is a limit point of A in $\langle X, d_1 \rangle$ iff x is a limit point of A in $\langle X, d_2 \rangle$.*
(2) *For any $A \subset X$, A is open in $\langle X, d_1 \rangle$ iff A is open in $\langle X, d_2 \rangle$.*
(3) *For any $A \subset X$, A is closed in $\langle X, d_1 \rangle$ iff A is closed in $\langle X, d_2 \rangle$.*
(4) *For any $x \in X$ and any $\varepsilon > 0$ there exists some $\delta > 0$ such that the open ball of radius δ centered at X in $\langle X, d_1 \rangle$ is a subset of the open ball of radius ε centered at X in $\langle X, d_2 \rangle$, and vice-versa (with d_1 and d_2 swapped).*

5.1 Definition. If metrics d_1 and d_2 for a set X are *equivalent* means that they satisfy the conditions of Theorem 39. (Sometimes such metrics are called *topologically equivalent.*)

Another way to make a "new" space out of an old one is to consider a subset of a metric space as a space in its own right. This is a very simple concept which can become much more confusing than you might expect. Technically, we first need a theorem to justify things.

Theorem 40. *Let $\langle X, d\rangle$ be a metric space and let $X_0 \subset X$. Let $d_0 : X_0 \times X_0 \to \mathbb{R}$ be the restriction of d to $X_0 \times X_0$ (i.e., $d_0(x, y) = d(x, y)$ for all $x, y \in X_0$). Then $\langle X_0, d_0\rangle$ is a metric space.*

5.2 Definition. The space $\langle X_0, d_0\rangle$ is called a *subspace* of $\langle X, d\rangle$, and when $X_0 \subset X$ and d_0 is the restriction of d as in Theorem 40. We will usually abuse notation slightly and use d for both of the metrics. When the metric is understood, we will say simply that X_0 is a subspace of X. For example, when we refer to $\mathbb{Q}$ as a subspace of $\mathbb{R}$, we mean the space $\langle \mathbb{Q}, d\rangle$ where $d(x, y) = |x - y|$ for $x, y \in \mathbb{Q}$.

Evidently, it will often happen that a given point is simultaneously a member of more than one space, and that a given set is simultaneously a subset of more than one space. This points up a deficiency in most of our definitions and notations, since they refer (in one way or another) to the space we are working in. Thus, "limit point of A" should be "limit point of A in $\langle X, d\rangle$," or simply "limit point of A in X if the metric is understood. When needed, we will use a subscript on the L operator: so $L_X(A)$ or $L_{\langle X,d\rangle}(A)$ denotes the corresponding set of all limit points, which is referred to as the set of limit points of A *relative* to X or to $\langle X, d\rangle$. For a simple example, let $X_0 = (0, 1]$ be a subspace of $\mathbb{R}$, and let $A = \{\, 1/n \mid n \in \mathbb{N} \,\}$ Then $L_{\mathbb{R}}(A) = \{0\}$, while $L_{X_0}(A) = L_{(0,1]} = \varnothing$. On the other hand, condition (1) of Theorem 39 can now be expressed as $L_{\langle X,d_1\rangle}(A) = L_{\langle X,d_2\rangle}(A)$ for all $A \subset X$.

Similarly, we need to consider the "relative" forms for the closure, interior, boundary, and exterior operators: $\mathrm{cl}_X(A)$, $\mathrm{int}_X(A)$, etc. For the open ball centered at a point, we have already used a subscript for the radius. We can either use a superscript for the space or we can move the radius inside the parenthesis:

$$B^X_\varepsilon(x) = B_X(x, \varepsilon) = \{\, y \in X \mid d(x, y) < \varepsilon \,\}$$

Finally, the notions of open and closed sets must be relativised: we will say that "A is open in X" or that "A is open relative to X" and similarly for closed sets. So, for the above example, the set A is not a closed subset in $\mathbb{R}$, but it *is* a closed set relative to $(0,1]$. Similarly $(1/2, 1]$ is a relatively open set in $(0,1]$ even though it is not open in $\mathbb{R}$.

Theorem 41. *Let X be a metric space, let $X_0 \subset X$ be a subspace of X, and let $A \subset X_0$. Then:*

(a) *A is open in X_0 iff $A = U \cap X_0$ for some $U \subset X$ such that U is open in X.*
(b) *A is closed in X_0 iff $A = U \cap X_0$ for some $U \subset X$ such that U is closed in X.*

You may have noticed that we did *not* relativise the definition of connected. This is because it is not really necessary.

Theorem 42. *Let $\langle X, d\rangle$ be metric space and let $A \subset X$. Then A is a connected set in the space $\langle X, d\rangle$ iff the subspace $\langle A, d\rangle$ of X is a connected metric space.*

Note that this means that Theorem 35 can be used to decide the connectedness of sets, by showing that there are no non-trivial sets which are relatively clopen.

§ **6** *MORE ON FUNCTIONS*

6.1 Definitions. A *1:1 correspondence* between sets X and Y is a 1:1 function f from X onto Y (i.e., $X = \mathrm{dom}(f)$ and $Y = \mathrm{ran}(f)$). When such an f exists, then f^{-1} is a 1:1 correspondence between Y and X, so we can simply say that X and Y are in 1:1 correspondence. A set is called *finite* if it is either empty or in 1:1 correspondence with a set of the form $\{1,\ldots,n\}$ where $n \in \mathbb{N}$. A set is *infinite* if it is not finite. A set is called *countably infinite* if it is in 1:1 correspondence with $\mathbb{N}$. A set is called *uncountable* if it is neither finite or countably infinite.

Theorem 43a (Cantor). *The sets ω, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite.*

Theorem 43b (Cantor). *The set $\mathbb{R}$ is uncountable.*

Theorem 44 (Schröder-Bernstein). *Let X and Y be sets. If X is in 1:1 correspondence with a subset of Y and Y is in 1:1 correspondence with a subset of X, then X and Y are in 1:1 correspondence.*

6.2 Definition. Let $f : X \to Y$. If $A \subset X$, then $f(A) \stackrel{\text{def}}{=} \{ f(x) \mid x \in A \}$ is called the *image of* A *under* f. If $B \subset Y$, then $f^{-1}(B) \stackrel{\text{def}}{=} \{ x \in X \mid f(x) \in B \}$ is called the *inverse image of* B *under* f.

Theorem 45. *Let* X *and* Y *be metric spaces and let* $f : X \to Y$. *The following conditions are equivalent:*

(1) *for each* $x \in X$ *and each positive number* ε *there exists a positive number* δ *such that* $f(B_X(x,\delta)) \subset B_Y(f(x),\varepsilon)$
(2) *if* $A \subset X$ *and* $x \in L_X(A)$, *then* $f(x) \in L_Y(f(A))$
(3) *if* $A \subset X$ *and* $x \in \mathrm{cl}_X(A)$, *then* $f(x) \in \mathrm{cl}_Y(f(A))$
(4) *if* U *is an open subset of* Y, *then* $f^{-1}(U)$ *is open in* X.
(5) *if* C *is an closed subset of* Y, *then* $f^{-1}(C)$ *is closed in* X.

6.3 Definition. Let X and Y be metric spaces and let $f : X \to Y$. If f satisfies the conditions of Theorem 45, we say that f is *continuous.* We also say that f is *continuous at* $x \in X$ when condition 1 of Theorem 45 is satisfied at that particular x. We say that f is a *homeomorphism* if it is a 1:1 correspondence from X onto Y and both f and f^{-1} are continuous. We say that X and Y are *homeomorphic* when a homeomorphism from X onto Y exists (note that f^{-1} is a homeomorphism from Y onto X, so this notations is justified). We also write $X \cong Y$ to denote that X and Y are homeomorphic.

Theorem 46. *If* $f : X \to Y$ *is continuous and* A *is a connected subset of* X, *then* $f(A)$ *is connected.*

Theorem 47. *If* $f : X \to Y$ *is a homeomorphism, then for any set* $A \subset X$, $f(L_X(A)) = L_Y(f(A))$ *and* $f(\mathrm{cl}_X(A)) = \mathrm{cl}_Y(f(A))$.

Theorem 48. *There exist non-homeomorphic metric spaces* X *and* Y *and a continuous 1:1 correspondence from* X *onto* Y.

Theorem 49. *Every non-empty open interval in* $\mathbb{R}$ *is homeomorphic to* $\mathbb{R}$.

Theorem 50. *Homeomorphism is an equivalence relation: i.e.,* $X \cong X$ *for any space* X; *if* $X \cong Y$ *then* $Y \cong X$; *and if* $X \cong Y$ *and* $Y \cong Z$ *then* $X \cong Z$.

Proposition 51. $\mathbb{N} \cong \mathbb{Z}$ *(as subspaces of* $\mathbb{R}$*).*

Proposition 52. $[0,1] \cong (0,1)$ *(as subspaces of* $\mathbb{R}$*).*

Proposition 53. $\mathbb{R} \cong \mathbb{P}$

Proposition 54. $\mathbb{P} \cong \mathbb{Q}$

Proposition 55. $\mathbb{Q} \cap [0,1] \cong \mathbb{Q} \cap (0,1)$

Theorem 56. *Let* X *be a set and let* d_1 *and* d_2 *be metrics for* X. *Then* d_1 *and* d_2 *are equivalent iff the identity function on* X *is a homeomorphism between* $\langle X, d_1 \rangle$ *and* $\langle X, d_2 \rangle$.

§ 7 *COVERS, COMPACTNESS, AND METRIC PRODUCTS*

7.1 Definition. A *cover* of a set A is a collection of sets whose union contains A. If $\mathcal{C}$ is a cover of A, then a *subcover* of $\mathcal{C}$ is a subcollection of $\mathcal{C}$ which is also a cover of A. (Technically, we should really say something like "subcover of $\mathcal{C}$ for A," but the set which is being covered will almost always be clear from the context.)

Proposition 57. *Work in* $\mathbb{R}$, *and let* I *be a non-trivial open interval (i.e.,* $I = (a,b)$ *where* $a < b$*). Then*

(a) *Any collection of open intervals which covers* I *has a subcover which is at most countable (i.e., which is either countably infinite or finite).*

(b) *Any collection of non-trivial closed intervals which covers* I *has a subcover which is at most countable ("non-trivial" again means* $[c,d]$ *where* $c < d$*)*

(c) *Any collection of open intervals which covers* I *has a subcover which is finite.*

(d) *Any collection of non-trivial closed intervals which covers* I *has a subcover which is finite.*

(e) *Any collection of open sets which covers* I *has a subcover which is at most countable.*

(f) *Any collection of infinite closed sets which covers* I *has a subcover which is at most countable.*

(g) *Any collection of open sets which covers* I *has a subcover which is finite.*

(h) *Any collection of infinite closed sets which covers* I *has a subcover which is finite.*

Proposition 58. *Work in $\mathbb{R}$, and let I be a non-trivial closed interval. Then each of the statements a–h of proposition 57 is true.*

7.2 Definition. In a metric space X, an *open cover* of a set A is a collection of open subsets of X which covers A. A set $A \subset X$ is *compact in* X means that every open cover of A has a finite subcover. The space X is *compact* means that it is a compact subset of itself.

Theorem 59. *If A is subset of a metric space X, then A is compact in X iff the subspace A of X is a compact space.*

It is important to note that 58-g is true—in other words, closed intervals in $\mathbb{R}$ *are* compact spaces. In fact, this generalizes.

Theorem 60. *A subset of $\mathbb{R}$ is compact iff it is closed and bounded.*

A small part of this characterization is true in any metric space.

Theorem 61. *If A is a compact subset of a metric space X, then A is closed in X.*

Note that Theorems 42 and 59 show that connectedness and compactness of a set are similar in that they don't depend upon which larger space the set "lives" in. Another similarity is that both these properties are preserved by continuous functions.

Theorem 62. *If $f : X \to Y$ is continuous and A is a compact subset of X, then $f(A)$ is compact.*

On the other hand, compactness is a bit "nicer" than connectedness in that it is preserved by both unions and intersections. (It is important to note that the empty set is, vacuously, compact.)

Theorem 63. *Let A and B be compact subsets of a metric space X. Then $A \cup B$ and $A \cap B$ are compact.*

Proposition 64. *Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of non-empty subsets of a metric space X such that $A_1 \supset A_2 \supset A_3 \supset \ldots$.*

(a) *If each A_i is compact, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact.*
(b) *If each A_i is compact and connected, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact and connected.*

7.3 Definition. Let X_1 and X_2 be metric spaces whose distance functions are d_1 and d_2 respectively. Then the function $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ defined by:

$$d(\langle x_1, x_2 \rangle , \langle y_1, y_2 \rangle) \stackrel{\text{def}}{=} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

is called the *product metric* on $X_1 \times X_2$.

Theorem 65. *The product metric from definition 6.3 is a metric on* $X_1 \times X_2$.

Unless otherwise stated, if we refer to $X_1 \times X_2$ as a metric space, then the metric on this product will be the product metric. Yet another similarity between connectedness and compactness is that they are both preserved by products.

Theorem 66. *Let* X_1 *and* X_2 *be connected metric spaces. Then* $X_1 \times X_2$ *is connected.*

Theorem 67. *Let* X_1 *and* X_2 *be compact metric spaces. Then* $X_1 \times X_2$ *is compact.*

As an application, we can now prove two of the most important foundational theorems from calculus: the intermediate and extreme value theorems. At this point we have the tools in place to make these theorems into just simple observations about connectedness and compactness.

Theorem 68. *Let* $I = [a, b]$ *be a non trivial closed interval and let* $f : I \to \mathbb{R}$ *be continuous. Then:*

(a) f *takes on all values between* $f(a)$ *and* $f(b)$. *More precisely, if* d *is a number such that either* $f(a) \le d \le f(b)$ *or* $f(b) \le d \le f(a)$, *then there is some* $c \in [a, b]$ *such that* $f(c) = d$.

(b) f *takes on a maximum and a minimum value. More precisely, there is some* $c \in [a, b]$ *such that* $f(c) \le f(x)$ *for all* $x \in [a, b]$ *and there is some* $c' \in [a, b]$ *such that* $f(c') \le f(x)$ *for all* $x \in [a, b]$.

To complete the association between Theorem 68 and the theorems in a standard calculus book, it is necessary to note what the assumption of the theorem really says. The interval I is considered to be a metric space, as a subspace of $\mathbb{R}$. So the continuity of f really says, in terms of the Calc-I definition of continuity, that f is continuous at each interior point of the interval and "right-hand" continuous at a and "left-hand" continuous at b. This is exactly how the intermediate and extreme value theorems are stated in Calc-I.

§ 8 *Topological Spaces*

The open sets in a metric space are what determine the limit points, connectedness, etc. In fact the open sets determine whether or not two metric spaces are homeomorphic, so they determine everything "topological" about a space. The following definition makes this more precise, and also generalizes the entire notion of a space.

8.1 Definition. A *topological space* is an ordered pair $\langle X, \mathcal{T} \rangle$, where X is a set and $\mathcal{T}$ is a collection of subsets of X which satisfy the following properties:

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$

(i) If U_1 and U_2 are sets in $\mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

(i) If $\mathcal{S}$ is a sub-collection of $\mathcal{T}$, then the union of $\mathcal{S}$ is a set which is in $\mathcal{T}$.

8.2 Notations. We refer to the sets in $\mathcal{T}$ as the *open* sets of the space $\langle X, \mathcal{T} \rangle$, and we refer to the collection $\mathcal{T}$ as the *topology* of the space. We will often refer to the space as X instead of $\langle X, \mathcal{T} \rangle$ in the same way that we have with metric spaces. We will also say that $\mathcal{T}$ is a *topology for* X whenever $\mathcal{T}$ is a collection of subsets of X which satisfy conditions (i), (ii), and (iii) of the above definition.

The next theorem justifies our use of the term "open set" for the members of a topology.

Theorem 69. *The collection of open subsets of a metric space $\langle X, d \rangle$ is, in fact, a topology for X.*

8.3 Definitions. The topology of theorem 69 is called the topology *generated by* the metric d. Note that two metrics are equivalent iff they generate the same topology.

A topological space $\langle X, \mathcal{T} \rangle$ is *metrizable* means that there is some metric d on X which generates the topology $\mathcal{T}$. We also refer to such a $\mathcal{T}$ as a *metrizable metric* for X.

Certainly, a metric spaces is a special kind of topological space. But most of the ideas we studied last semester will generalize in a natural way to this new setting. Usually, we can just replace "every $\varepsilon > 0$" with "every open set." For example:

1.1t Definition. If X is a topological space, $A \subset X$, and $p \in X$, then p *is a limit point of* A means that every open subset of X which contains p contains a point which is in A and is different from p.

Obviously, once limit point is defined we have a definition for closed set in a topological space. Similarly, most (but not all) of the definitions in sections 1–7 can be generalized to this new setting. For example, Definition 4.1t defines connectedness, but we can't simply form Definition 7.3t to define product spaces. Once the definitions generalize, so do most (but not all) of the theorems and propositions.

8.4 Exercise. Decide which of the definitions, theorems, and propositions from sections 1–7 have a corresponding statement which is suffixed by "t." Prove or disprove each of the corresponding theorems and propositions. Do any of the theorems become false statements?

Here are the two simplest examples of topological spaces:

8.5 Examples. For any set X, the collection of *all* subsets of X forms a topology for X. This topology is called the *discrete* topology for X, and it is generated by the discrete metric. On the other hand, the collection $\{\varnothing, X\}$ is also a topology for X, which is called the *indiscrete* topology for X. (Note that correct spelling is important; a topological space might get insulted if you say that it is indiscreet!)

Theorem 70. *If X is a set with more than one point, then the indiscrete topology for X is not metrizable. There is also a non-metrizable topology for X which is not the indiscrete topology.*

In many ways the indiscrete topology is just too messy to be very useful. In a sense, the topology doesn't even realize that the space has more than one point in it! One way to express this is to consider the the following conditions:

8.6 Definition. Let X be a topological space. Then X satisfies the "axiom" T_n (where n equals 0, 1, or 2) means that whenever x_1 and x_2 are points of X and $x_1 \neq x_2$ there exist open subsets U_1 and U_2 of X such that $x_1 \in U_1$ and $x_2 \in U_2$ and:

T_0 : either $x_1 \notin U_2$ or $x_2 \notin U_1$.
T_1 : $x_1 \notin U_2$ and $x_2 \notin U_1$.
T_2 : $U_1 \cap U_2 = \varnothing$

A space which satisfies axiom T_2 is also called a *Hausdorff* space.

Theorem 71. *A topological space X is T_1 iff every single-point subset of X is a closed set in X.*

Theorem 72. *If X is a finite set, then the only T_1 topology on X is the discrete topology.*

8.7 Definition. Let X be a T_1 topological space. Then X is T_3 or *regular* means that if $x \in X$ and C is a closed subset of X which does not contain x, then there exist open subsets U_1 and U_2 of X such that $x \in U_1$, $C \subset U_2$, and $U_1 \cap U_2 = \varnothing$. X is T_4 or *normal* means that if C_1 and C_2 is are closed subset of X which are disjoint, then there exist open subsets U_1 and U_2 of X such that $C_1 \subset U_1$, $C \subset U_2$, and $U_1 \cap U_2 = \varnothing$.

Note that one could also consider the definitions in 8.7 without the requirement that X is T_1. But the resulting conditions don't imply each other in the nice hierarchical way that ours do, i.e., it follows directly from the definitions and Theorem 71 that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$.

Theorem 73. *For each $n = 0, 1, 2, 3$ there is an example of a topological space which is T_n and is not T_{n+1}.*

Since the union of a large collection of finite sets might very well be infinite, we can't directly use the collection of all finite subsets of a set X to define a topology for X.

Theorem 74. *If X is a set and $\mathcal{T}$ consists of all subsets of X which are either empty or co-finite in X, then $\mathcal{T}$ is a topology on X.* (A subset A of X is *co-finite* in X means that $X \setminus A$ is finite.

Theorem 75. *If X is a set and $\mathcal{T}$ consists of all subsets of X which are either empty or co-countable in X, then $\mathcal{T}$ is a topology on X.* (A subset A of X is *co-countable* in X means that $X \setminus A$ is either finite or countable.

Section 5 on subspaces should have been a minor stumbling point for Exercise 8.4. Hopefully you used the following natural definition.

Theorem 76. *Let $\langle X, \mathcal{T} \rangle$ be a topological space, let $A \subset X$, and let $\mathcal{T}' = \{ A \cap U : U \in \mathcal{T} \}$. Then $\langle A, \mathcal{T}' \rangle$ is a topological space.*

8.8 Definition. The topology $\mathcal{T}'$ in theorem 76 is called the *subspace topology on A*, and we refer to A as a *subspace of X* when we are using this topology.

Proposition 77. *For $n = 0, 1, 2, 3, 4$, every subspace of a T_n space is a T_n space.*

Theorem 78. *Every metrizable space is normal (T_4).*

§ 9 *TOPOLOGICAL BASES*

Sometimes, describing an entire topology can get complicated. The notion of a "base" for a topology can make life much easier.

9.1 Definition. Let X be a set. The collection $\mathcal{B}$ of subsets of X is a *topological base for* X means that:

(i) Every point of X is in some member of $\mathcal{B}$, i.e., $X = \bigcup \mathcal{B}$

(ii) If U_1 and U_2 are sets in $\mathcal{B}$ and $x \in U_1 \cap U_2$ then there is some $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U_1 \cap U_2$.

Theorem 79. *If $\mathcal{B}$ is a topological base for X and $\mathcal{T} \stackrel{\text{def}}{=} \{ \bigcup \mathcal{C} : \mathcal{C} \subset \mathcal{B} \}$, then $\mathcal{T}$ is a topology for X (i.e., $\langle X, \mathcal{T} \rangle$ is a topological space).*

9.2 Notations. The topology in Theorem 79 is called the topology *generated by* $\mathcal{B}$. If $\mathcal{B}$ generates the topology $\mathcal{T}$ on X, we say that $\mathcal{B}$ is a *base for* $\mathcal{T}$ (or a base for X when the topology is understood from the context). It may happen that different topological bases for the same set generate the same topology—in this case we say that the bases are *equivalent.*

Theorem 80. *Let d be a metric for X. Then $\{ B_\varepsilon(x) : x \in X, \varepsilon > 0 \}$ is a topological base for X which generates the same topology that d does.*

Theorem 81. *Let $\mathcal{B}$ be a topological base for X, and let X have the topology generated by $\mathcal{B}$. Then a subset U of X is open iff for each $x \in U$ there is some $B \in \mathcal{B}$ such that $x \in B \subset U$.*

If $\mathcal{B}$ is a base for the topology of X, then *almost* any statement which involves the phrase "every open subset" will be equivalent to the statement obtained by replacing that phrase with "every member of $\mathcal{B}$." Here are a few examples:

Theorem 82. *Let X be a topological space and let $\mathcal{B}$ be base for X. Then*

(1) *$x \in L(A)$ iff every member of $\mathcal{B}$ which contains x contains a point of A which is different from x.*

(2) *$x \in \mathrm{cl}(A)$ iff every member of $\mathcal{B}$ which contains x intersects A.*

(3) *X is compact iff every subcollection of $\mathcal{B}$ which covers X has a finite subcover.*

(4) *$f : Y \to X$ is continuous iff $f^{-1}(B)$ is open in Y for every $B \in \mathcal{B}$.*

January 27, 2000

Theorem 83. *Let $\mathcal{B}$ be a base for the topological space X and let A be a subspace of X. Then $\{ B \cap A : B \in \mathcal{B} \}$ is a base for A.*

The notion of base can also be "localized."

9.3 Definition. Let $x \in X$ where X is a topological space, and let $\mathcal{B}_x$ be a collection open subsets of X all of which contain x. Then $\mathcal{B}_x$ is a *local base for* x means that for every open set U containing x there exists a set $B \in \mathcal{B}_x$ such that $B \subset U$. We sometimes use the phrase *local base for x in X* to make it clear what space we are considering x to be in.

We'll do more with local bases in the next section, but one easy fact is the following.

Theorem 84. *If $\mathcal{B}$ is a topological base for X and $x \in X$, then $\{ B \in \mathcal{B} : x \in B \}$ is a local base for x. If, for each $x \in X$, $\mathcal{B}_x$ is a local base for x, then $\bigcup \{ \mathcal{B}_x : x \in X \}$ is a base for X.*

Theorem 85. *The collection $\{ [a, b) : a, b \in \mathbb{R} \}$ is a topological base for $\mathbb{R}$. The topology generated by this base is not metrizable.*

Theorem 86. *The collection $\{ (a, b) : a, b \in \mathbb{R} \} \cup \{ \{x\} : x \in \mathbb{P} \}$ is a topological base for $\mathbb{R}$. The topology generated by this base is not metrizable.*

The space of Theorem 85 is called the *Sorgenfrey line,* the space of Theorem 86 is called the *Michael line.*

Proposition 87. *The Sorgenfrey and Michael lines are T_i for $i = 0, 1, 2, 3, 4$.*

The notion of a product space should have been another stumbling block in Exercise 8.4. The problem is that if $\langle X_1, \mathcal{T}_1 \rangle$ and $\langle X_2, \mathcal{T}_2 \rangle$ are topological spaces, then $\{ U_1 \times U_2 : U_1 \in \mathcal{T}_1,\ U_2 \in \mathcal{T}_2 \}$ will probably not be a topology on $X_1 \times X_2$ (this happens even for $\mathbb{R} \times \mathbb{R}$). However, this collection does form a base for the "correct" topology on the product.

Theorem 88. *Let X_1 and X_2 be topological spaces with bases $\mathcal{B}_1$ and B_2 respectively. Then $\{ B_1 \times B_2 : B_1 \in \mathcal{B}_1,\ B_2 \in \mathcal{B}_2 \}$ is a topological base for $X_1 \times X_2$. The resulting topology is independent of the bases chosen for X_1 and X_2.*

9.4 Definition. The topology of Theorem 88 is called the *product topology.* Unless another topology is explicitly stated, we assume that this is the topology on the product of a pair of topological spaces.

Theorem 90. *Let* $\langle X_1, d_1 \rangle$ *and* $\langle X_2, d_2 \rangle$ *be metric spaces. If we give each* X_i *the topology generated by its metric, then the product topology on* $X_1 \times X_2$ *is the same as the topology generated by the product metric obtained from* d_1 *and* d_2.

Proposition 91. *If* $n \in \{0, 1, 2, 3, 4\}$ *and* X *and* Y *are topological spaces which are both* T_n, *then* $X \times Y$ *is* T_n.

§ **10** *COUNTABILITY CONDITIONS*

10.1 Convention. For the remainder of these notes we will take the term "countable" to mean "either empty, finite, or countably infinite." This will save us the trouble of writing "at most countable" over and over again. While this convention is quite common, you should be aware that it is *not* universally accepted. Whenever you see the term "countable" used elsewhere, you should be aware that it might be used in this way, but it also might mean "countably infinite."

10.2 Definitions. A space is *second countable* means that it has a countable base. A space X is *first countable at a point* $x \in X$ means that there exists a countable local base for x in X. A space is *first countable* means that it is first countable at each of its points.

10.3 Definitions. A subset D of a set A (in a topological space) is *dense in* A means that $\mathrm{cl}(D) \supset A$. A space X is *separable* means that there is a countable subset of X which is dense in X. A space is Lindelöf means that every open cover of X has a countable subcover.

Theorem 92. *Every metrizable space is first countable.*

Theorem 93. *A metrizable space is separable iff it is second countable.*

Proposition 94. *Every first countable space is metrizable.*

Proposition 95. *The Sorgenfrey and Michael lines are Lindelöf.*

Theorem 96. *Every compact Hausdorff space is normal.*

Theorem 97. *Every regular Lindelöf space is normal.*

The following theorem was accidentally omitted from section 7. It should have come right after Theorem 60.

Theorem 98. *Let X be a compact space, and let $A \subset X$. Then A is compact iff A is closed in X.*

§ **11** *LIMITS OF SEQUENCES*

11.4 Notation. *Let*

February 22, 2000

Theorem 96. *Every compact Hausdorff space is normal.*

Theorem 97. *Every regular Lindelöf space is normal.*

The following theorem was accidentally omitted from section 7. It should have come right after Theorem 60.

Theorem 98. *Let X be a compact space, and let $A \subset X$. Then A is compact iff A is closed in X.*

Another important characterization of compactness is the following.

Theorem 99. *A space X is compact iff $\bigcap \mathcal{C} \neq \varnothing$ for every collection $\mathcal{C}$ of closed sets which has the finite intersection property.* A collection $\mathcal{C}$ of sets has the *finite intersection property* means that the intersection of every non-empty finite subcollection of $\mathcal{C}$ is non-empty.

§ **11** *LIMITS OF SEQUENCES*

11.1 Definition. An *infinite sequence* is a function σ whose domain is a set of the form $\text{dom}(\sigma) = \{\, i \in \mathbb{Z} : i \geq N \,\}$, where $N \in \mathbb{Z}$ is fixed. The most common cases are $N = 1$ and $N = 0$ which correspond, respectively, to $\text{dom}(\sigma) = \mathbb{N}$ and $\text{dom}(\sigma) = \omega$. When the range of a sequence σ is a subset of a set X, we say that σ is an *infinite sequence in X*.

For notational simplicity, we will only refer to sequences whose domain is $\mathbb{N}$ in these notes. The generalization of all the concepts to the case of other domains will be obvious.

11.2 Notations. When σ is a sequence, it is a common practice to use subscripts rather than functional notation to denote the values of the function, i.e., $\sigma_i \stackrel{\text{def}}{=} \sigma(i)$. Since there are only countably many values, once can often just list them in a way that makes the pattern clear, such as

$$\sigma = \langle 1, 1/2, 1/3, 1/4, \ldots \rangle$$

which is just another way of saying $\sigma(i) \stackrel{\text{def}}{=} 1/i$ or $\sigma_i \stackrel{\text{def}}{=} 1/i$. Sometimes the angle brackets are omitted, or even worse, set braces are used instead. In other classes, you may have seen "variable notation" used, with different symbols used for the sequence itself and the values, such as letting σ denote the sequence defined by $a_i = 1/i$. Often, the whole sequence is then denoted by something like $\{ a_i \}_{i=1}^{\infty}$, which is not very good since it should really denote the range of the sequence, not the function itself. A better notion is $\langle a_i \rangle_{i=1}^{\infty}$. In these notes, we will mostly stick with functional notation.

11.3 Definition. Let X be a topological space, let $\sigma : \mathbb{N} \to X$ be an infinite sequence in X, and let $p \in X$. Then p is a *limit of* σ means that for every open set U containing p there is an $N \in \mathbb{N}$ such that $\sigma(i) \in U$ for all $i \geq N$. Alternatively, we say that σ *converges to* p, and we say that σ is *convergent* when it has a limit. We often write $\lim \sigma = p$ or $\lim_{i\to\infty} \sigma(i) = p$ or $\lim_{i\to\infty} \sigma_i = p$ to denote that p is a limit of σ.

The following three propositions are true in metrizable spaces. They are false in general topological spaces. Try to find the best (i.e., least restrictive) condition on the topology of X which will make them true.

Proposition 100. *Limits are unique if they exist. I.e., if σ is an infinite sequence in a topological space and p_1 and p_2 are both limits of σ, then $p_1 = p_2$.*

When proposition 99 holds true for X, we can refer to p as *the limit* of σ. One should never use any of the "$\lim \sigma =$" notations unless the space being studied satisfies proposition 99.

Proposition 101. *Let X be a topological space and let $A \subset X$. Then $p \in \mathrm{cl}(A)$ iff there is an infinite sequence in A which converges to p. Also, $p \in L(A)$ iff there is an infinite sequence σ in A which converges to p such that $\sigma(i) \neq p$ for all i.*

Proposition 102. *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then f is continuous iff f preserves limits of sequences, i.e., whenever σ is a sequence in X which converges to p, $f \circ \sigma$ converges to $f(p)$.*

Note that $f \circ \sigma$ is just the sequence $\langle f(\sigma_1), f(\sigma_2), \ldots \rangle$.

11.4 Definition. Let σ be in infinite sequence. Then τ is a *subsequence* of σ means that $\tau = \sigma \circ \eta$, where $\eta : \mathbb{N} \to \mathbb{N}$ satisfies the condition that $\eta(i) > \eta(j)$ whenever $i > j$. We often write $\tau_i = \sigma_{\eta_i}$ instead of $\tau(i) = \sigma(\eta(i))$ to denote this.

Proposition 102. *If σ is an infinite sequence in a topological which converges to p, then every subsequence of σ also converges to p.*

11.5 Definition. A space X is *sequentially compact* means that every infinite sequence in X has a convergent subsequence.

Theorem 103. *A metrizable space is sequentially compact iff it is compact.*

Theorem 104. *There is a T_4 space which is compact but is not sequentially compact.*

Theorem 104. *There is a T_4 space which is sequentially compact but is not compact.*

March 23, 2000

Theorem 103. *A metrizable space is sequentially compact iff it is compact.*

Theorem 104. *There is a T_4 space which is compact but is not sequentially compact.*

Theorem 104. *There is a T_4 space which is sequentially compact but is not compact.*

There is more than one way to express the idea that the topology of a space is "determined" by the convergence of sequences. The difference between the two definitions below is actually rather subtle.

11.6 Definition. A topological space X is *Frechet* means that for every $A \subset X$ and $p \in X$, $p \in \mathrm{cl}(A)$ iff there is an infinite sequence in A which converges to p (i.e., X satisfies Proposition 101). A topological space X is *sequential* means that for every $A \subset X$, A is closed iff for every infinite sequence σ in A, if p is a limit of σ then $p \in A$.

Theorem 105. *Every Frechet space is sequential.*

Theorem 106. *There exists a space which is not sequential.*

Theorem 107. *There exists a sequential space which is not Frechet.*

Infinite sequences play a particularly important role in metric spaces.

11.7 Definition. Let $\langle X, d\rangle$ be a metric space. An infinite sequence σ in X is a *Cauchy* sequence means that for every $\varepsilon > 0$ there is an integer $N \in \mathbb{N}$ such that $d(\sigma(i), \sigma(j)) < \varepsilon$ whenever i and j are bigger than N. A metric space X is *complete* means that every Cauchy sequence in X is convergent. A metrizable topological space X is *completely metrizable* if there is a complete metric for X which generates the topology of X.

Theorem 108. *Every compact metric space is complete.*

Theorem 109. *A subspace of a complete metric space is complete iff it is closed.*

Theorem 110. *The product of two complete metric spaces is a complete metric space.*

Theorem 111. $\mathbb{R}$ *is a complete metric space.*

Theorem 112. $\mathbb{Q}$ *is not a complete metric space.*

In fact, $\mathbb{Q}$ is not a completely metrizable topological space.

Theorem 113. *Let X be an infinite topological space with no isolated points.* (A point $x \in X$ is called *isolated* if $\{x\}$ is an open subset of X.) *If X is completely metrizable, then X is uncountable.*

On the other hand, the following is rather surprising.

Theorem 114. $\mathbb{P}$ *is a completely metrizable topological space.*

A final important fact is that every metric space can be "completed."

Theorem 115. *Every metric space is a subspace of a complete metric space.*

12.1 Definition. Let $\langle X_1, X_2, \ldots \rangle$ be an infinite sequence of sets. Then the set of infinite sequences

$$\prod_{i=1}^{\infty} X_i \stackrel{\text{def}}{=} \big\{\langle x_1, x_2, \ldots \rangle \,:\, x_i \in X_i \text{ for all } i \in \mathbb{N}\big\}$$

is called the *Cartesian product* of $\langle X_1, X_2, \ldots \rangle$.

Theorem 116. *Let $\langle X_1, X_2, \ldots \rangle$ be an infinite sequence of topological spaces, and let $X = \prod_{i=1}^{\infty} X_i$. Define families $\mathcal{B}_\Box$ and $\mathcal{B}_T$ of subsets of X by:*

$$\mathcal{B}_\Box \stackrel{\text{def}}{=} \big\{\textstyle\prod_{i=1}^{\infty} B_i \,:\, B_i \text{ is an open subset of } X_i\big\}$$
$$\mathcal{B}_T \stackrel{\text{def}}{=} \big\{\textstyle\prod_{i=1}^{\infty} B_i \in \mathcal{B}_\Box \,:\, \{\, i \in \mathbb{N} : B_i \neq X_i \,\}\text{is finite}\big\}$$

Then $B_\Box$ and B_T are both topological bases for X.

12.2 Definition. The topology generated by $B_\Box$ is called the *box topology*, and when $\prod_{i=1}^{\infty} X_i$ is given this topology, we call it the *box product* of the spaces involved. Similarly, B_T generates a topology called the *Tychonoff topology*, and we refer to the resulting space as the *Tychonoff* product.

From a simple point of view, the box product seems to be the more straightforward and natural topology. However, in the majority of applications in topology use the Tychonoff topology instead. One reason for this are the following theorems.

Theorem 117 Tychonoff's Theorem. *The Tychonoff product of a infinite sequence of compact spaces is a compact space.*

Theorem 118. *The Tychonoff product of an infinite sequence of connected spaces is a connected space.*

Theorem 119. *The Tychonoff product of an infinite sequence of metrizable spaces is a metrizable space.*

Proposition 120. *Let X be the box product of $\langle X_1, X_2, \ldots \rangle$ where each $X_i = [0,1]$ (and $[0,1]$ has the usual subspace of $\mathbb{R}$). Then X is neither compact, connected, nor metrizable.*

March 23, 2000

Theorem 118. *The Tychonoff product of an infinite sequence of connected spaces is a connected space.*

Theorem 119. *The Tychonoff product of an infinite sequence of metrizable spaces is a metrizable space.*

Proposition 120. *Let X be the box product of $\langle X_1, X_2, \ldots \rangle$ where each $X_i = [0,1]$ (and $[0,1]$ has the usual subspace of $\mathbb{R}$). Then X is neither compact, connected, nor metrizable.*

§ **13** *CONTINUA*

13.1 Definition. A *continuum* is a compact and connected T_2 space. A metrizable continuum is, obviously, a continuum which is metrizable. A planar continuum is a continuum which is homeomorphic to a subspace of $\mathbb{R}^2$.

Proposition 121. *Let X be a T_2 space and let $X_1, X_2, X_3, \ldots$ is an infinite sequence of subsets of X such that $X_1 \supset X_2 \supset X_3 \supset \ldots$.*

a) *If each X_i is non-empty and compact then $\bigcap_{i=1}^{\infty} X_i$ is non-empty and compact.*
b) *If each X_i is a non-empty continuum then $\bigcap_{i=1}^{\infty} X_i$ is a non-empty continuum.*
c) *If each X_i is connected then $\bigcap_{i=1}^{\infty} X_i$ is connected.*

Note that condition (c) is just a re-statement for proposition 37. It is included here for emphasis. Statement (b) is, in fact, true and it is one of the major devices for defining complicated continua.

13.2 Definitions. If X is a continuum, then a *subcontinuum* of X is simply a subspace of X which is a continuum. A *proper subcontinuum* of X is a subcontinuum of X which is not equal to X. A continuum is *irreducible* between points x and y means that no proper subcontinuum contains both x and y. A continuum is *decomposable* means that it is equal the union of two of its proper subcontinua. A continuum which is not decomposable is called *indecomposable.*

Theorem 122.

a) $[0,1]$ *is irreducible between* 0 *and* 1.

b) *Let* $X \subset \mathbb{R}^2$ *be defined by:* $X = \{ \langle x, y \rangle : x \in (0,1], y = sin(\pi/x) \} \bigcup \{ \langle x, y \rangle : x = 0, y \in [-1,1] \}$. *Then* X *is a continuum and* X *is irreducible between* $\langle 1, 0 \rangle$ *and any of its points on the* y*-axis.*

Theorem 123. *There exists a plane continuum* X *which contains three distinct points such that* X *is irreducible between any two of these three points.*

The continuum for Theorem 123 cannot be a "garden variety" example. To see this, first note that virtually every continuum you know about is, in fact, decomposable. The interval is since $[0,1] = [0,0.9] \cup [0.1,1]$. It is easy to see that example (b) of Theorem 122 is also decomposable. It should not be obvious that indecomposable continua even exist. However:

Theorem 124. *A continuum* X *is indecomposable iff* X *contains three distinct points such that* X *is irreducible between any two of these three points.*

13.3 Definition. Let X be a topological space and fix $x \in X$. Then the *connected component of* X *containing* x is the union of all the connected subsets of X which contain x.

Theorem 125. *Let* X *be a* T_2 *space, and for each* $x \in X$, *let* C_x *denote the connected component of* X *containing* x. *Then each* C_x *is a closed set, and the collection* $\{ C_x : x \in X \}$ *is a partition of* X *(i.e., this collection is pairwise disjoint and its union equals* X*).*

Theorem 126. *Let* X *be a continuum and let* U *be an open subset of* X. *Then every component of* $X \setminus U$ *intersects the boundary of* U.

Theorem 127. *If a continuum contains more than one point, then it must be uncountable.*

Theorem 128. *There exist infinite, countable, connected, Hausdorff spaces.*

13.4 Definition. The *composant* of a point x in a continuum X is the union of all the proper subcontinua of X which contain x.

Theorem 129. *If* X *is an infinite continuum which is indecomposable, then the collection of all composants of* X *forms a partition of* X. *Furthermore, there are uncountably many distinct composants.*