Corrections: pages 28 and 29. There are two each of propositions 102 and 104. Re-number them 102a, 102b, 104a, and 104b.

Theorem 118. *The Tychonoff product of an infinite sequence of connected spaces is a connected space.*

Theorem 119. *The Tychonoff product of an infinite sequence of metrizable spaces is a metrizable space.*

Proposition 120. *Let X be the box product of $\langle X_1, X_2, \ldots \rangle$ where each $X_i = [0,1]$ (and $[0,1]$ has the usual subspace of $\mathbb{R}$). Then X is neither compact, connected, nor metrizable.*

§ 13 *CONTINUA*

13.1 Definition. A *continuum* is a compact and connected T_2 space. A metrizable continuum is, obviously, a continuum which is metrizable. A planar continuum is a continuum which is homeomorphic to a subspace of $\mathbb{R}^2$.

Proposition 121. *Let X be a T_2 space and let $X_1, X_2, X_3, \ldots$ is an infinite sequence of subsets of X such that $X_1 \supset X_2 \supset X_3 \supset \ldots$.*

a) *If each X_i is non-empty and compact then $\bigcap_{i=1}^{\infty} X_i$ is non-empty and compact.*
b) *If each X_i is a non-empty continuum then $\bigcap_{i=1}^{\infty} X_i$ is a non-empty continuum.*
c) *If each X_i is connected then $\bigcap_{i=1}^{\infty} X_i$ is connected.*

Note that condition (c) is just a re-statement for proposition 37. It is included here for emphasis. Statement (b) is, in fact, true and it is one of the major devices for defining complicated continua.

13.2 Definitions. If X is a continuum, then a *subcontinuum* of X is simply a subspace of X which is a continuum. A *proper subcontinuum* of X is a subcontinuum of X which is not equal to X. A continuum is *irreducible* between points x and y means that no proper subcontinuum contains both x and y. A continuum is *decomposable* means that it is equal the union of two of its proper subcontinua. A continuum which is not decomposable is called *indecomposable.*

Theorem 122.

a) $[0,1]$ *is irreducible between* 0 *and* 1.

b) *Let* $X \subset \mathbb{R}^2$ *be defined by:* $X = \{\langle x,y\rangle : x \in (0,1], y = sin(\pi/x)\} \bigcup \{\langle x,y\rangle : x = 0, y \in [-1,1]\}$. *Then* X *is a continuum and* X *is irreducible between* $\langle 1,0\rangle$ *and any of its points on the* y*-axis.*

Theorem 123. *There exists a plane continuum* X *which contains three distinct points such that* X *is irreducible between any two of these three points.*

The continuum for Theorem 123 cannot be a "garden variety" example. To see this, first note that virtually every continuum you know about is, in fact, decomposable. The interval is since $[0,1] = [0,0.9] \cup [0.1,1]$. It is easy to see that example (b) of Theorem 122 is also decomposable. It should not be obvious that indecomposable continua even exist. However:

Theorem 124. *A continuum* X *is indecomposable iff* X *contains three distinct points such that* X *is irreducible between any two of these three points.*

13.3 Definition. Let X be a topological space and fix $x \in X$. Then the *connected component of* X *containing* x is the union of all the connected subsets of X which contain x.

Theorem 125. *Let* X *be a* T_2 *space, and for each* $x \in X$, *let* C_x *denote the connected component of* X *containing* x. *Then each* C_x *is a closed set, and the collection* $\{C_x : x \in X\}$ *is a partition of* X *(i.e., this collection is pairwise disjoint and its union equals* X*).*

Theorem 126. *Let* X *be a continuum and let* U *be an open subset of* X. *Then every component of* $X \setminus U$ *intersects the boundary of* U.

Theorem 127. *If a continuum contains more than one point, then it must be uncountable.*

Theorem 128. *There exist infinite, countable, connected, Hausdorff spaces.*

13.4 Definition. The *composant* of a point x in a continuum X is the union of all the proper subcontinua of X which contain x.

Theorem 129. *If* X *is an infinite continuum which is indecomposable, then the collection of all composants of* X *forms a partition of* X. *Furthermore, there are uncountably many distinct composants.*