Theorem 103. *A metrizable space is sequentially compact iff it is compact.*

Theorem 104. *There is a T_4 space which is compact but is not sequentially compact.*

Theorem 104. *There is a T_4 space which is sequentially compact but is not compact.*

There is more than one way to express the idea that the topology of a space is "determined" by the convergence of sequences. The difference between the two definitions below is actually rather subtle.

11.6 Definition. A topological space X is *Frechet* means that for every $A \subset X$ and $p \in X$, $p \in \mathrm{cl}(A)$ iff there is an infinite sequence in A which converges to p (i.e., X satisfies Proposition 101). A topological space X is *sequential* means that for every $A \subset X$, A is closed iff for every infinite sequence σ in A, if p is a limit of σ then $p \in A$.

Theorem 105. *Every Frechet space is sequential.*

Theorem 106. *There exists a space which is not sequential.*

Theorem 107. *There exists a sequential space which is not Frechet.*

Infinite sequences play a particularly important role in metric spaces.

11.7 Definition. Let $\langle X, d\rangle$ be a metric space. An infinite sequence σ in X is a *Cauchy* sequence means that for every $\varepsilon > 0$ there is an integer $N \in \mathbb{N}$ such that $d(\sigma(i), \sigma(j)) < \varepsilon$ whenever i and j are bigger than N. A metric space X is *complete* means that every Cauchy sequence in X is convergent. A metrizable topological space X is *completely metrizable* if there is a complete metric for X which generates the topology of X.

Theorem 108. *Every compact metric space is complete.*

Theorem 109. *A subspace of a complete metric space is complete iff it is closed.*

Theorem 110. *The product of two complete metric spaces is a complete metric space.*

Theorem 111. $\mathbb{R}$ *is a complete metric space.*

Theorem 112. $\mathbb{Q}$ *is not a complete metric space.*

In fact, $\mathbb{Q}$ is not a completely metrizable topological space.

Theorem 113. *Let X be an infinite topological space with no isolated points.* (A point $x \in X$ is called *isolated* if $\{x\}$ is an open subset of X.) *If X is completely metrizable, then X is uncountable.*

On the other hand, the following is rather surprising.

Theorem 114. $\mathbb{P}$ *is a completely metrizable topological space.*

A final important fact is that every metric space can be "completed."

Theorem 115. *Every metric space is a subspace of a complete metric space.*

§ **12** *INFINITE PRODUCTS*

12.1 Definition. Let $\langle X_1, X_2, \ldots \rangle$ be an infinite sequence of sets. Then the set of infinite sequences

$$\prod_{i=1}^{\infty} X_i \stackrel{\text{def}}{=} \left\{ \langle x_1, x_2, \ldots \rangle \,:\, x_i \in X_i \text{ for all } i \in \mathbb{N} \right\}$$

is called the *Cartesian product* of $\langle X_1, X_2, \ldots \rangle$.

Theorem 116. *Let $\langle X_1, X_2, \ldots \rangle$ be an infinite sequence of topological spaces, and let $X = \prod_{i=1}^{\infty} X_i$. Define families $\mathcal{B}_{\square}$ and $\mathcal{B}_T$ of subsets of X by:*

$$\mathcal{B}_{\square} \stackrel{\text{def}}{=} \left\{ \textstyle\prod_{i=1}^{\infty} B_i \,:\, B_i \text{ is an open subset of } X_i \right\}$$
$$\mathcal{B}_T \stackrel{\text{def}}{=} \left\{ \textstyle\prod_{i=1}^{\infty} B_i \in \mathcal{B}_{\square} \,:\, \{\, i \in \mathbb{N} : B_i \neq X_i \,\} \text{is finite} \right\}$$

Then $B_{\square}$ and B_T are both topological bases for X.

12.2 Definition. The topology generated by $B_{\square}$ is called the *box topology*, and when $\prod_{i=1}^{\infty} X_i$ is given this topology, we call it the *box product* of the spaces involved. Similarly, B_T generates a topology called the *Tychonoff topology*, and we refer to the resulting space as the *Tychonoff* product.

From a simple point of view, the box product seems to be the more straightforward and natural topology. However, in the majority of applications in topology use the Tychonoff topology instead. One reason for this are the following theorems.

Theorem 117 Tychonoff's Theorem. *The Tychonoff product of a infinite sequence of compact spaces is a compact space.*