

Theorem 96. *Every compact Hausdorff space is normal.*

Theorem 97. *Every regular Lindelöf space is normal.*

The following theorem was accidentally omitted from section 7. It should have come right after Theorem 60.

Theorem 98. *Let X be a compact space, and let $A \subset X$. Then A is compact iff A is closed in X .*

Another important characterization of compactness is the following.

Theorem 99. *A space X is compact iff $\bigcap \mathcal{C} \neq \emptyset$ for every collection \mathcal{C} of closed sets which has the finite intersection property.* A collection \mathcal{C} of sets has the *finite intersection property* means that the intersection of every non-empty finite subcollection of \mathcal{C} is non-empty.

§ 11 LIMITS OF SEQUENCES

11.1 Definition. An *infinite sequence* is a function σ whose domain is a set of the form $\text{dom}(\sigma) = \{i \in \mathbb{Z} : i \geq N\}$, where $N \in \mathbb{Z}$ is fixed. The most common cases are $N = 1$ and $N = 0$ which correspond, respectively, to $\text{dom}(\sigma) = \mathbb{N}$ and $\text{dom}(\sigma) = \omega$. When the range of a sequence σ is a subset of a set X , we say that σ is an *infinite sequence in X* .

For notational simplicity, we will only refer to sequences whose domain is \mathbb{N} in these notes. The generalization of all the concepts to the case of other domains will be obvious.

11.2 Notations. When σ is a sequence, it is a common practice to use subscripts rather than functional notation to denote the values of the function, i.e., $\sigma_i \stackrel{\text{def}}{=} \sigma(i)$. Since there are only countably many values, one can often just list them in a way that makes the pattern clear, such as

$$\sigma = \langle 1, 1/2, 1/3, 1/4, \dots \rangle$$

which is just another way of saying $\sigma(i) \stackrel{\text{def}}{=} 1/i$ or $\sigma_i \stackrel{\text{def}}{=} 1/i$. Sometimes the angle brackets are omitted, or even worse, set braces are used instead. In other classes, you may have seen “variable notation” used, with different symbols used for the sequence itself and the values, such as letting σ denote the sequence defined by $a_i = 1/i$. Often, the whole sequence is then denoted by something like $\{a_i\}_{i=1}^{\infty}$, which is not very good since it should really

denote the range of the sequence, not the function itself. A better notion is $\langle a_i \rangle_{i=1}^{\infty}$. In these notes, we will mostly stick with functional notation.

11.3 Definition. Let X be a topological space, let $\sigma : \mathbb{N} \rightarrow X$ be an infinite sequence in X , and let $p \in X$. Then p is a *limit of σ* means that for every open set U containing p there is an $N \in \mathbb{N}$ such that $\sigma(i) \in U$ for all $i \geq N$. Alternatively, we say that σ *converges to p* , and we say that σ is *convergent* when it has a limit. We often write $\lim \sigma = p$ or $\lim_{i \rightarrow \infty} \sigma(i) = p$ or $\lim_{i \rightarrow \infty} \sigma_i = p$ to denote that p is a limit of σ .

The following three propositions are true in metrizable spaces. They are false in general topological spaces. Try to find the best (i.e., least restrictive) condition on the topology of X which will make them true.

Proposition 100. *Limits are unique if they exist. I.e., if σ is an infinite sequence in a topological space and p_1 and p_2 are both limits of σ , then $p_1 = p_2$.*

When proposition 99 holds true for X , we can refer to p as *the limit* of σ . One should never use any of the “ $\lim \sigma =$ ” notations unless the space being studied satisfies proposition 99.

Proposition 101. *Let X be a topological space and let $A \subset X$. Then $p \in \text{cl}(A)$ iff there is an infinite sequence in A which converges to p . Also, $p \in L(A)$ iff there is an infinite sequence σ in A which converges to p such that $\sigma(i) \neq p$ for all i .*

Proposition 102. *Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a function. Then f is continuous iff f preserves limits of sequences, i.e., whenever σ is a sequence in X which converges to p , $f \circ \sigma$ converges to $f(p)$.*

Note that $f \circ \sigma$ is just the sequence $\langle f(\sigma_1), f(\sigma_2), \dots \rangle$.

11.4 Definition. Let σ be an infinite sequence. Then τ is a *subsequence* of σ means that $\tau = \sigma \circ \eta$, where $\eta : \mathbb{N} \rightarrow \mathbb{N}$ satisfies the condition that $\eta(i) > \eta(j)$ whenever $i > j$. We often write $\tau_i = \sigma_{\eta_i}$ instead of $\tau(i) = \sigma(\eta(i))$ to denote this.

Proposition 102. *If σ is an infinite sequence in a topological space which converges to p , then every subsequence of σ also converges to p .*

11.5 Definition. A space X is *sequentially compact* means that every infinite sequence in X has a convergent subsequence.

Theorem 103. *A metrizable space is sequentially compact iff it is compact.*

Theorem 104. *There is a T_4 space which is compact but is not sequentially compact.*

Theorem 104. *There is a T_4 space which is sequentially compact but is not compact.*