

Corrections: 1) Page 23. Proposition 77 should start “For $n = 0, 1, 2, 3, 4,$ ”

2) The previous page (containing the start of section 9) should be numbered page 24.

Theorem 83. *Let \mathcal{B} be a base for the topological space X and let A be a subspace of X . Then $\{B \cap A : B \in \mathcal{B}\}$ is a base for A .*

The notion of base can also be “localized.”

9.3 Definition. Let $x \in X$ where X is a topological space, and let \mathcal{B}_x be a collection open subsets of X all of which contain x . Then \mathcal{B}_x is a *local base for x* means that for every open set U containing x there exists a set $B \in \mathcal{B}_x$ such that $B \subset U$. We sometimes use the phrase *local base for x in X* to make it clear what space we are considering x to be in.

We’ll do more with local bases in the next section, but one easy fact is the following.

Theorem 84. *If \mathcal{B} is a topological base for X and $x \in X$, then $\{B \in \mathcal{B} : x \in B\}$ is a local base for x . If, for each $x \in X$, \mathcal{B}_x is a local base for x , then $\bigcup\{\mathcal{B}_x : x \in X\}$ is a base for X .*

Theorem 85. *The collection $\{(a, b) : a, b \in \mathbb{R}\}$ is a topological base for \mathbb{R} . The topology generated by this base is not metrizable.*

Theorem 86. *The collection $\{(a, b) : a, b \in \mathbb{R}\} \cup \{\{x\} : x \in \mathbb{P}\}$ is a topological base for \mathbb{R} . The topology generated by this base is not metrizable.*

The space of Theorem 85 is called the *Sorgenfrey line*, the space of Theorem 86 is called the *Michael line*.

Proposition 87. *The Sorgenfrey and Michael lines are T_i for $i = 0, 1, 2, 3, 4$.*

The notion of a product space should have been another stumbling block in Exercise 8.4. The problem is that if $\langle X_1, \mathcal{T}_1 \rangle$ and $\langle X_2, \mathcal{T}_2 \rangle$ are topological spaces, then $\{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{T}_2\}$ will probably not be a topology on $X_1 \times X_2$ (this happens even for $\mathbb{R} \times \mathbb{R}$). However, this collection does form a base for the “correct” topology on the product.

Theorem 88. *Let X_1 and X_2 be topological spaces with bases \mathcal{B}_1 and \mathcal{B}_2 respectively. Then $\{B_1 \times B_2 : B_1 \in \mathcal{B}_1, B_2 \in \mathcal{B}_2\}$ is a topological base for $X_1 \times X_2$. The resulting topology is independent of the bases chosen for X_1 and X_2 .*

9.4 Definition. The topology of Theorem 88 is called the *product topology*. Unless another topology is explicitly stated, we assume that this is the topology on the product of a pair of topological spaces.

Theorem 90. *Let $\langle X_1, d_1 \rangle$ and $\langle X_2, d_2 \rangle$ be metric spaces. If we give each X_i the topology generated by its metric, then the product topology on $X_1 \times X_2$ is the same as the topology generated by the product metric obtained from d_1 and d_2 .*

Proposition 91. *If $n \in \{0, 1, 2, 3, 4\}$ and X and Y are topological spaces which are both T_n , then $X \times Y$ is T_n .*

§ 10 COUNTABILITY CONDITIONS

10.1 Convention. For the remainder of these notes we will take the term “countable” to mean “either empty, finite, or countably infinite.” This will save us the trouble of writing “at most countable” over and over again. While this convention is quite common, you should be aware that it is *not* universally accepted. Whenever you see the term “countable” used elsewhere, you should be aware that it might be used in this way, but it also might mean “countably infinite.”

10.2 Definitions. A space is *second countable* means that it has a countable base. A space X is *first countable at a point* $x \in X$ means that there exists a countable local base for x in X . A space is *first countable* means that it is first countable at each of its points.

10.3 Definitions. A subset D of a set A (in a topological space) is *dense in* A means that $\text{cl}(D) \supset A$. A space X is *separable* means that there is a countable subset of X which is dense in X . A space is *Lindelöf* means that every open cover of X has a countable subcover.

Theorem 92. *Every metrizable space is first countable.*

Theorem 93. *A metrizable space is separable iff it is second countable.*

Proposition 94. *Every first countable space is metrizable.*

Proposition 95. *The Sorgenfrey and Michael lines are Lindelöf.*

Theorem 96. *Every compact Hausdorff space is normal.*

Theorem 97. *Every regular Lindelöf space is normal.*

The following theorem was accidentally omitted from section 7. It should have come right after Theorem 60.

Theorem 98. *Let X be a compact space, and let $A \subset X$. Then A is compact iff A is closed in X .*

§ 11 LIMITS OF SEQUENCES