Theorem 78. *Every metrizable space is normal (T_4).*

§ **9** *TOPOLOGICAL BASES*

Sometimes, describing an entire topology can get complicated. The notion of a "base" for a topology can make life much easier.

9.1 Definition. Let X be a set. The collection $\mathcal{B}$ of subsets of X is a *topological base for* X means that:

(i) Every point of X is in some member of $\mathcal{B}$, i.e., $X = \bigcup \mathcal{B}$

(ii) If U_1 and U_2 are sets in $\mathcal{B}$ and $x \in U_1 \cap U_2$ then there is some $V \in \mathcal{B}$ such that $x \in V$ and $V \subset U_1 \cap U_2$.

Theorem 79. *If $\mathcal{B}$ is a topological base for X and $\mathcal{T} \stackrel{\text{def}}{=} \{\bigcup \mathcal{C} : \mathcal{C} \subset \mathcal{B}\}$, then $\mathcal{T}$ is a topology for X (i.e., $\langle X, \mathcal{T} \rangle$ is a topological space).*

9.2 Notations. The topology in Theorem 79 is called the topology *generated by* $\mathcal{B}$. If $\mathcal{B}$ generates the topology $\mathcal{T}$ on X, we say that $\mathcal{B}$ is a *base for* $\mathcal{T}$ (or a base for X when the topology is understood from the context). It may happen that different topological bases for the same set generate the same topology—in this case we say that the bases are *equivalent.*

Theorem 80. *Let d be a metric for X. Then $\{B_\varepsilon(x) : x \in X, \varepsilon > 0\}$ is a topological base for X which generates the same topology that d does.*

Theorem 81. *Let $\mathcal{B}$ be a topological base for X, and let X have the topology generated by $\mathcal{B}$. Then a subset U of X is open iff for each $x \in U$ there is some $B \in \mathcal{B}$ such that $x \in B \subset U$.*

If $\mathcal{B}$ is a base for the topology of X, then *almost* any statement which involves the phrase "every open subset" will be equivalent to the statement obtained by replacing that phrase with "every member of $\mathcal{B}$." Here are a few examples:

Theorem 82. *Let X be a topological space and let $\mathcal{B}$ be base for X. Then*

(1) *$x \in L(A)$ iff every member of $\mathcal{B}$ which contains x contains a point of A which is different from x.*

(2) *$x \in \mathrm{cl}(A)$ iff every member of $\mathcal{B}$ which contains x intersects A.*

(3) *X is compact iff every subcollection of $\mathcal{B}$ which covers X has a finite subcover.*

(4) *$f : Y \to X$ is continuous iff $f^{-1}(B)$ is open in Y for every $B \in \mathcal{B}$.*