Corrections: 1) Pages 16–20. This is section 7, not 6. Re-number definitions 6.1 thru 6.3 on these pages as 7.1 thru 7.3. 2) Page 15. Change $\cap_X(A)$ to $\operatorname{int}_X(A)$ in the second paragraph.

§ **8** *Topological Spaces*

The open sets in a metric space are what determine the limit points, connectedness, etc. In fact the open sets determine whether or not two metric spaces are homeomorphic, so they determine everything "topological" about a space. The following definition makes this more precise, and also generalizes the entire notion of a space.

8.1 Definition. A *topological space* is an ordered pair $\langle X, \mathcal{T} \rangle$, where X is a set and $\mathcal{T}$ is a collection of subsets of X which satisfy the following properties:

(i) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$

(i) If U_1 and U_2 are sets in $\mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

(i) If $\mathcal{S}$ is a sub-collection of $\mathcal{T}$, then the union of $\mathcal{S}$ is a set which is in $\mathcal{T}$.

8.2 Notations. We refer to the sets in $\mathcal{T}$ as the *open* sets of the space $\langle X, \mathcal{T} \rangle$, and we refer to the collection $\mathcal{T}$ as the *topology* of the space. We will often refer to the space as X instead of $\langle X, \mathcal{T} \rangle$ in the same way that we have with metric spaces. We will also say that $\mathcal{T}$ is a *topology for* X whenever $\mathcal{T}$ is a collection of subsets of X which satisfy conditions (i), (ii), and (iii) of the above definition.

The next theorem justifies our use of the term "open set" for the members of a topology.

Theorem 69. *The collection of open subsets of a metric space $\langle X, d \rangle$ is, in fact, a topology for X.*

8.3 Definitions. The topology of theorem 69 is called the topology *generated by* the metric d. Note that two metrics are equivalent iff they generate the same topology.

A topological space $\langle X, \mathcal{T} \rangle$ is *metrizable* means that there is some metric d on X which generates the topology $\mathcal{T}$. We also refer to such a $\mathcal{T}$ as a *metrizable metric* for X.

Certainly, a metric spaces is a special kind of topological space. But most of the ideas we studied last semester will generalize in a natural way to this new setting. Usually, we can just replace "every $\varepsilon > 0$" with "every open set." For example:

1.1t Definition. If X is a topological space, $A \subset X$, and $p \in X$, then *p is a limit point of A* means that every open subset of X which contains p contains a point which is in A and is different from p.

Obviously, once limit point is defined we have a definition for closed set in a topological space. Similarly, most (but not all) of the definitions in sections 1–7 can be generalized to this new setting. For example, Definition 4.1t defines connectedness, but we can't simply form Definition 7.3t to define product spaces. Once the definitions generalize, so do most (but not all) of the theorems and propositions.

8.4 Exercise. Decide which of the definitions, theorems, and propositions from sections 1–7 have a corresponding statement which is suffixed by "t." Prove or disprove each of the corresponding theorems and propositions. Do any of the theorems become false statements?

Here are the two simplest examples of topological spaces:

8.5 Examples. For any set X, the collection of *all* subsets of X forms a topology for X. This topology is called the *discrete* topology for X, and it is generated by the discrete metric. On the other hand, the collection $\{\varnothing, X\}$ is also a topology for X, which is called the *indiscrete* topology for X. (Note that correct spelling is important; a topological space might get insulted if you say that it is indiscreet!)

Theorem 70. *If X is a set with more than one point, then the indiscrete topology for X is not metrizable. There is also a non-metrizable topology for X which is not the indiscrete topology.*

In many ways the indiscrete topology is just too messy to be very useful. In a sense, the topology doesn't even realize that the space has more than one point in it! One way to express this is to consider the the following conditions:

8.6 Definition. Let X be a topological space. Then X satisfies the "axiom" T_n (where n equals 0, 1, or 2) means that whenever x_1 and x_2 are points of X and $x_1 \neq x_2$ there exist open subsets U_1 and U_2 of X such that $x_1 \in U_1$ and $x_2 \in U_2$ and:

T_0 : either $x_1 \notin U_2$ or $x_2 \notin U_1$.
T_1 : $x_1 \notin U_2$ and $x_2 \notin U_1$.
T_2 : $U_1 \cap U_2 = \varnothing$

A space which satisfies axiom T_2 is also called a *Hausdorff* space.

Theorem 71. *A topological space X is T_1 iff every single-point subset of X is a closed set in X.*

Theorem 72. *If X is a finite set, then the only T_1 topology on X is the discrete topology.*

8.7 Definition. Let X be a T_1 topological space. Then X is T_3 or *regular* means that if $x \in X$ and C is a closed subset of X which does not contain x, then there exist open subsets U_1 and U_2 of X such that $x \in U_1$, $C \subset U_2$, and $U_1 \cap U_2 = \varnothing$. X is T_4 or *normal* means that if C_1 and C_2 is are closed subset of X which are disjoint, then there exist open subsets U_1 and U_2 of X such that $C_1 \subset U_1$, $C \subset U_2$, and $U_1 \cap U_2 = \varnothing$.

Note that one could also consider the definitions in 8.7 without the requirement that X is T_1. But the resulting conditions don't imply each other in the nice hierarchical way that ours do, i.e., it follows directly from the definitions and Theorem 71 that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$.

Theorem 73. *For each $n = 0, 1, 2, 3$ there is an example of a topological space which is T_n and is not T_{n+1}.*

Since the union of a large collection of finite sets might very well be infinite, we can't directly use the collection of all finite subsets of a set X to define a topology for X.

Theorem 74. *If X is a set and $\mathcal{T}$ consists of all subsets of X which are either empty or co-finite in X, then $\mathcal{T}$ is a topology on X.* (A subset A of X is *co-finite* in X means that $X \setminus A$ is finite.

Theorem 75. *If X is a set and $\mathcal{T}$ consists of all subsets of X which are either empty or co-countable in X, then $\mathcal{T}$ is a topology on X.* (A subset A of X is *co-countable* in X means that $X \setminus A$ is either finite or countable.

Section 5 on subspaces should have been a minor stumbling point for Exercise 8.4. Hopefully you used the following natural definition.

Theorem 76. *Let $\langle X, \mathcal{T} \rangle$ be a topological space, let $A \subset X$, and let $\mathcal{T}' = \{ A \cap U : U \in \mathcal{T} \}$. Then $\langle A, \mathcal{T}' \rangle$ is a topological space.*

8.8 Definition. The topology $\mathcal{T}'$ in theorem 76 is called the *subspace topology on A*, and we refer to A as a *subspace of X* when we are using this topology.

Proposition 77. *For $n = 0, 1, 2, 3, 4$, every subspace of a T_n space is a T_n space.*