Correction. Page 16: re-number Theorem 43 as 43a, and the first Theorem 44 as 44b.
Clarification. Page 13: Change the end of Proposition 37 to read “$\bigcap_{i=1}^{\infty} A_i$ is connected if it is non-empty.”

§ 7 COVERS, COMPACTNESS, AND METRIC PRODUCTS

7.1 Definition. A cover of a set $A$ is a collection of sets whose union contains $A$. If $C$ is a cover of $A$, then a subcover of $C$ is a subcollection of $C$ which is also a cover of $A$. (Technically, we should really say something like “subcover of $C$ for $A$,” but the set which is being covered will almost always be clear from the context.)

Proposition 57. Work in $\mathbb{R}$, and let $I$ be a non-trivial open interval (i.e., $I = (a, b)$ where $a < b$). Then

(a) Any collection of open intervals which covers $I$ has a subcover which is at most countable (i.e., which is either countably infinite or finite).
(b) Any collection of non-trivial closed intervals which covers $I$ has a subcover which is at most countable (“non-trivial” again means $[c, d]$ where $c < d$)
(c) Any collection of open intervals which covers $I$ has a subcover which is infinite.
(d) Any collection of non-trivial closed intervals which covers $I$ has a subcover which is finite.
(e) Any collection of open sets which covers $I$ has a subcover which is at most countable.
(f) Any collection of infinite closed sets which covers $I$ has a subcover which is at most countable.
(g) Any collection of open sets which covers $I$ has a subcover which is finite.
(h) Any collection of infinite closed sets which covers $I$ has a subcover which is finite.

Proposition 58. Work in $\mathbb{R}$, and let $I$ be a non-trivial closed interval. Then each of the statements a–h of proposition 57 is true.

7.2 Definition. In a metric space $X$, an open cover of a set $A$ is a collection of open subsets of $X$ which covers $A$. A set $A \subset X$ is compact in $X$ means that every open cover of $A$ has a finite subcover. The space $X$ is compact means that it is a compact subset of itself.
Theorem 59. If $A$ is subset of a metric space $X$, then $A$ is compact in $X$ iff the subspace $A$ of $X$ is a compact space.

It is important to note that 58-g is true—in other words, closed intervals in $\mathbb{R}$ are compact spaces. In fact, this generalizes.

Theorem 60. A subset of $\mathbb{R}$ is compact iff it is closed and bounded.

A small part of this characterization is true in any metric space.

Theorem 61. If $A$ is a compact subset of a metric space $X$, then $A$ is closed in $X$.

Note that Theorems 42 and 59 show that connectedness and compactness of a set are similar in that they don’t depend upon which larger space the set “lives” in. Another similarity is that both these properties are preserved by continuous functions.

Theorem 62. If $f : X \to Y$ is continuous and $A$ is a compact subset of $X$, then $f(A)$ is compact.

On the other hand, compactness is a bit “nicer” than connectedness in that it is preserved by both unions and intersections. (It is important to note that the empty set is, vacuously, compact.)

Theorem 63. Let $A$ and $B$ be compact subsets of a metric space $X$. Then $A \cup B$ and $A \cap B$ are compact.

Proposition 64. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of non-empty subsets of a metric space $X$ such that $A_1 \supset A_2 \supset A_3 \supset \ldots$.

(a) If each $A_i$ is compact, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact.

(b) If each $A_i$ is compact and connected, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact and connected.

7.3 Definition. Let $X_1$ and $X_2$ be metric spaces whose distance functions are $d_1$ and $d_2$ respectively. Then the function $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ defined by:

$$d((x_1, x_2), (y_1, y_2)) \overset{\text{def}}{=} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

is called the product metric on $X_1 \times X_2$.

Theorem 65. The product metric from definition 6.3 is a metric on $X_1 \times X_2$. 

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Unless otherwise stated, if we refer to \( X_1 \times X_2 \) as a metric space, then the metric on this product will be the product metric. Yet another similarity between connectedness and compactness is that they are both preserved by products.

**Theorem 66.** Let \( X_1 \) and \( X_2 \) be connected metric spaces. Then \( X_1 \times X_2 \) is connected.

**Theorem 67.** Let \( X_1 \) and \( X_2 \) be compact metric spaces. Then \( X_1 \times X_2 \) is compact.

As an application, we can now prove two of the most important foundational theorems from calculus: the intermediate and extreme value theorems. At this point we have the tools in place to make these theorems into just simple observations about connectedness and compactness.

**Theorem 68.** Let \( I = [a,b] \) be a non trivial closed interval and let \( f : I \to \mathbb{R} \) be continuous. Then:

(a) \( f \) takes on all values between \( f(a) \) and \( f(b) \). More precisely, if \( d \) is a number such that either \( f(a) \leq d \leq f(b) \) or \( f(b) \leq d \leq f(a) \), then there is some \( c \in [a,b] \) such that \( f(c) = d \).

(b) \( f \) takes on a maximum and a minimum value. More precisely, there is some \( c \in [a,b] \) such that \( f(c) \leq f(x) \) for all \( x \in [a,b] \) and there is some \( c' \in [a,b] \) such that \( f(c') \leq f(x) \) for all \( x \in [a,b] \).

To complete the association between Theorem 68 and the theorems in a standard calculus book, it is necessary to note what the assumption of the theorem really says. The interval \( I \) is considered to be a metric space, as a subspace of \( \mathbb{R} \). So the continuity of \( f \) really says, in terms of the Calc-I definition of continuity, that \( f \) is continuous at each interior point of the interval and “right-hand” continuous at \( a \) and “left-hand” continuous at \( b \). This is exactly how the intermediate and extreme value theorems are stated in Calc-I.