Correction. *Page 16: re-number Theorem 43 as 43a, and the first Theorem 44 as 44b.*

Clarification. *Page 13: Change the end of Proposition 37 to read "$\bigcap_{i=1}^{\infty} A_i$ is connected if it is non-empty."*

§ 7 COVERS, COMPACTNESS, AND METRIC PRODUCTS

7.1 Definition. A *cover* of a set A is a collection of sets whose union contains A. If $\mathcal{C}$ is a cover of A, then a *subcover* of $\mathcal{C}$ is a subcollection of $\mathcal{C}$ which is also a cover of A. (Technically, we should really say something like "subcover of $\mathcal{C}$ for A," but the set which is being covered will almost always be clear from the context.)

Proposition 57. *Work in $\mathbb{R}$, and let I be a non-trivial open interval (i.e., $I = (a, b)$ where $a < b$). Then*

(a) *Any collection of open intervals which covers I has a subcover which is at most countable (i.e., which is either countably infinite or finite).*

(b) *Any collection of non-trivial closed intervals which covers I has a subcover which is at most countable ("non-trivial" again means $[c, d]$ where $c < d$)*

(c) *Any collection of open intervals which covers I has a subcover which is finite.*

(d) *Any collection of non-trivial closed intervals which covers I has a subcover which is finite.*

(e) *Any collection of open sets which covers I has a subcover which is at most countable.*

(f) *Any collection of infinite closed sets which covers I has a subcover which is at most countable.*

(g) *Any collection of open sets which covers I has a subcover which is finite.*

(h) *Any collection of infinite closed sets which covers I has a subcover which is finite.*

Proposition 58. *Work in $\mathbb{R}$, and let I be a non-trivial closed interval. Then each of the statements a–h of proposition 57 is true.*

7.2 Definition. In a metric space X, an *open cover* of a set A is a collection of open subsets of X which covers A. A set $A \subset X$ is *compact in* X means that every open cover of A has a finite subcover. The space X is *compact* means that it is a compact subset of itself.

Theorem 59. *If A is subset of a metric space X, then A is compact in X iff the subspace A of X is a compact space.*

It is important to note that 58-g is true—in other words, closed intervals in $\mathbb{R}$ *are* compact spaces. In fact, this generalizes.

Theorem 60. *A subset of $\mathbb{R}$ is compact iff it is closed and bounded.*

A small part of this characterization is true in any metric space.

Theorem 61. *If A is a compact subset of a metric space X, then A is closed in X.*

Note that Theorems 42 and 59 show that connectedness and compactness of a set are similar in that they don't depend upon which larger space the set "lives" in. Another similarity is that both these properties are preserved by continuous functions.

Theorem 62. *If $f : X \to Y$ is continuous and A is a compact subset of X, then $f(A)$ is compact.*

On the other hand, compactness is a bit "nicer" than connectedness in that it is preserved by both unions and intersections. (It is important to note that the empty set is, vacuously, compact.)

Theorem 63. *Let A and B be compact subsets of a metric space X. Then $A \cup B$ and $A \cap B$ are compact.*

Proposition 64. *Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of non-empty subsets of a metric space X such that $A_1 \supset A_2 \supset A_3 \supset \ldots$.*

(a) *If each A_i is compact, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact.*

(b) *If each A_i is compact and connected, then $\bigcap_{i=1}^{\infty} A_i$ is non-empty and compact and connected.*

7.3 Definition. Let X_1 and X_2 be metric spaces whose distance functions are d_1 and d_2 respectively. Then the function $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$ defined by:

$$d(\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle) \stackrel{\text{def}}{=} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

is called the *product metric* on $X_1 \times X_2$.

Theorem 65. *The product metric from definition 6.3 is a metric on $X_1 \times X_2$.*

Unless otherwise stated, if we refer to $X_1 \times X_2$ as a metric space, then the metric on this product will be the product metric. Yet another similarity between connectedness and compactness is that they are both preserved by products.

Theorem 66. *Let X_1 and X_2 be connected metric spaces. Then $X_1 \times X_2$ is connected.*

Theorem 67. *Let X_1 and X_2 be compact metric spaces. Then $X_1 \times X_2$ is compact.*

As an application, we can now prove two of the most important foundational theorems from calculus: the intermediate and extreme value theorems. At this point we have the tools in place to make these theorems into just simple observations about connectedness and compactness.

Theorem 68. *Let $I = [a, b]$ be a non trivial closed interval and let $f : I \to \mathbb{R}$ be continuous. Then:*

(a) *f takes on all values between $f(a)$ and $f(b)$. More precisely, if d is a number such that either $f(a) \le d \le f(b)$ or $f(b) \le d \le f(a)$, then there is some $c \in [a, b]$ such that $f(c) = d$.*

(b) *f takes on a maximum and a minimum value. More precisely, there is some $c \in [a, b]$ such that $f(c) \le f(x)$ for all $x \in [a, b]$ and there is some $c' \in [a, b]$ such that $f(c') \le f(x)$ for all $x \in [a, b]$.*

To complete the association between Theorem 68 and the theorems in a standard calculus book, it is necessary to note what the assumption of the theorem really says. The interval I is considered to be a metric space, as a subspace of $\mathbb{R}$. So the continuity of f really says, in terms of the Calc-I definition of continuity, that f is continuous at each interior point of the interval and "right-hand" continuous at a and "left-hand" continuous at b. This is exactly how the intermediate and extreme value theorems are stated in Calc-I.