§ **5** *RELATIVISATION*

Obviously a given set X can have more than one metric (e.g., Proposition 14). This is one way of creating a "new" metric space out of an old one. But, sometimes the difference between two metrics on a set is not "topological," in that it doesn't effect the limit points, etc.

Theorem 39. *Let X be a set and let d_1 and d_2 be metrics on X. The following conditions are equivalent:*

(1) *For any $A \subset X$ and any $x \in X$, x is a limit point of A in $\langle X, d_1\rangle$ iff x is a limit point of A in $\langle X, d_2\rangle$.*
(2) *For any $A \subset X$, A is open in $\langle X, d_1\rangle$ iff A is open in $\langle X, d_2\rangle$.*
(3) *For any $A \subset X$, A is closed in $\langle X, d_1\rangle$ iff A is closed in $\langle X, d_2\rangle$.*
(4) *For any $x \in X$ and any $\varepsilon > 0$ there exists some $\delta > 0$ such that the open ball of radius δ centered at X in $\langle X, d_1\rangle$ is a subset of the open ball of radius ε centered at X in $\langle X, d_2\rangle$, and vice-versa (with d_1 and d_2 swapped).*

5.1 Definition. If metrics d_1 and d_2 for a set X are *equivalent* means that they satisfy the conditions of Theorem 39. (Sometimes such metrics are called *topologically equivalent.*)

Another way to make a "new" space out of an old one is to consider a subset of a metric space as a space in its own right. This is a very simple concept which can become much more confusing than you might expect. Technically, we first need a theorem to justify things.

Theorem 40. *Let $\langle X, d\rangle$ be a metric space and let $X_0 \subset X$. Let $d_0 : X_0 \times X_0 \to \mathbb{R}$ be the restriction of d to $X_0 \times X_0$ (i.e., $d_0(x,y) = d(x,y)$ for all $x, y \in X_0$). Then $\langle X_0, d_0\rangle$ is a metric space.*

5.2 Definition. The space $\langle X_0, d_0\rangle$ is called a *subspace* of $\langle X, d\rangle$, and when $X_0 \subset X$ and d_0 is the restriction of d as in Theorem 40. We will usually abuse notation slightly and use d for both of the metrics. When the metric is understood, we will say simply that X_0 is a subspace of X. For example, when we refer to $\mathbb{Q}$ as a subspace of $\mathbb{R}$, we mean the space $\langle \mathbb{Q}, d\rangle$ where $d(x,y) = |x - y|$ for $x, y \in \mathbb{Q}$.

Evidently, it will often happen that a given point is simultaneously a member of more than one space, and that a given set is simultaneously a subset of more than one space. This points up a deficiency in most of our definitions and notations, since they refer (in one way or another) to the space we are working in. Thus, "limit point of A" should be "limit point of A in $\langle X, d\rangle$," or simply "limit point of A in X if the metric is understood. When needed, we will use a subscript on the L operator: so $L_X(A)$ or $L_{\langle X,d\rangle}(A)$ denotes the corresponding set of all limit points, which is referred to as the set of limit points of A *relative* to X or to $\langle X, d\rangle$. For a simple example, let $X_0 = (0, 1]$ be a subspace of $\mathbb{R}$, and let $A = \{\, 1/n \mid n \in \mathbb{N} \,\}$ Then $L_{\mathbb{R}}(A) = \{0\}$, while $L_{X_0}(A) = L_{(0,1]} = \varnothing$. On the other hand, condition (1) of Theorem 39 can now be expressed as $L_{\langle X,d_1\rangle}(A) = L_{\langle X,d_2\rangle}(A)$ for all $A \subset X$.

Similarly, we need to consider the "relative" forms for the closure, interior, boundary, and exterior operators: $\mathrm{cl}_X(A)$, $\cap_X(A)$, etc. For the open ball centered at a point, we have already used a subscript for the radius. We can either use a superscript for the space or we can move the radius inside the parenthesis:

$$B^X_\varepsilon(x) = B_X(x, \varepsilon) = \{\, y \in X \mid d(x, y) < \varepsilon \,\}$$

Finally, the notions of open and closed sets must be relativised: we will say that "A is open in X" or that "A is open relative to X" and similarly for closed sets. So, for the above example, the set A is not a closed subset in $\mathbb{R}$, but it *is* a closed set relative to $(0, 1]$. Similarly $(1/2, 1]$ is a relatively open set in $(0, 1]$ even though it is not open in $\mathbb{R}$.

Theorem 41. *Let X be a metric space, let $X_0 \subset X$ be a subspace of X, and let $A \subset X_0$. Then:*

(a) *A is open in X_0 iff $A = U \cap X_0$ for some $U \subset X$ such that U is open in X.*
(b) *A is closed in X_0 iff $A = U \cap X_0$ for some $U \subset X$ such that U is closed in X.*

You may have noticed that we did *not* relativise the definition of connected. This is because it is not really necessary.

Theorem 42. *Let $\langle X, d\rangle$ be metric space and let $A \subset X$. Then A is a connected set in the space $\langle X, d\rangle$ iff the subspace $\langle A, d\rangle$ of X is a connected metric space.*

Note that this means that Theorem 35 can be used to decide the connectedness of sets, by showing that there are no non-trivial sets which are relatively clopen.

6.1 Definitions. A *1:1 correspondence* between sets X and Y is a 1:1 function f from X onto Y (i.e., $X = \mathrm{dom}(f)$ and $Y = \mathrm{ran}(f)$). When such an f exists, then f^{-1} is a 1:1 correspondence between Y and X, so we can simply say that X and Y are in 1:1 correspondence. A set is called *finite* if it is either empty or in 1:1 correspondence with a set of the form $\{1, \dots, n\}$ where $n \in \mathbb{N}$. A set is *infinite* if it is not finite. A set is called *countably infinite* if it is in 1:1 correspondence with $\mathbb{N}$. A set is called *uncountable* if it is neither finite or countably infinite.

Theorem 43 (Cantor). *The sets ω, $\mathbb{Z}$, and $\mathbb{Q}$ are all countably infinite.*

Theorem 44 (Cantor). *The set $\mathbb{R}$ is uncountable.*

Theorem 44 (Schröder-Bernstein). *Let X and Y be sets. If X is in 1:1 correspondence with a subset of Y and Y is in 1:1 correspondence with a subset of X, then X and Y are in 1:1 correspondence.*

6.2 Definition. Let $f : X \to Y$. If $A \subset X$, then $f(A) \overset{\text{def}}{=} \{\, f(x) \mid x \in A \,\}$ is called the *image of A under f*. If $B \subset Y$, then $f^{-1}(B) \overset{\text{def}}{=} \{\, x \in X \mid f(x) \in B \,\}$ is called the *inverse image of B under f*.

Theorem 45. *Let X and Y be metric spaces and let $f : X \to Y$. The following conditions are equivalent:*

1. *for each $x \in X$ and each positive number ε there exists a positive number δ such that $f(B_X(x,\delta)) \subset B_Y(f(x),\varepsilon)$*
2. *if $A \subset X$ and $x \in L_X(A)$, then $f(x) \in L_Y(f(A))$*
3. *if $A \subset X$ and $x \in \mathrm{cl}_X(A)$, then $f(x) \in \mathrm{cl}_Y(f(A))$*
4. *if U is an open subset of Y, then $f^{-1}(U)$ is open in X.*
5. *if C is an closed subset of Y, then $f^{-1}(C)$ is closed in X.*

6.3 Definition. Let X and Y be metric spaces and let $f : X \to Y$. If f satisfies the conditions of Theorem 45, we say that f is *continuous.* We also say that f is *continuous at $x \in X$* when condition 1 of Theorem 45 is satisfied at that particular x. We say that f is a *homeomorphism* if it is a 1:1 correspondence from X onto Y and both f and f^{-1} are continuous. We say that X and Y are *homeomorphic* when a homeomorphism from X

onto Y exists (note that f^{-1} is a homeomorphism from Y onto X, so this notations is justified). We also write $X \cong Y$ to denote that X and Y are homeomorphic.

Theorem 46. *If* $f : X \to Y$ *is continuous and* A *is a connected subset of* X*, then* $f(A)$ *is connected.*

Theorem 47. *If* $f : X \to Y$ *is a homeomorphism, then for any set* $A \subset X$*,* $f(L_X(A)) = L_Y(f(A))$ *and* $f(\mathrm{cl}_X(A)) = \mathrm{cl}_Y(f(A))$*.*

Theorem 48. *There exist non-homeomorphic metric spaces* X *and* Y *and a continuous 1:1 correspondence from* X *onto* Y*.*

Theorem 49. *Every non-empty open interval in* $\mathbb{R}$ *is homeomorphic to* $\mathbb{R}$*.*

Theorem 50. *Homeomorphism is an equivalence relation: i.e.,* $X \cong X$ *for any space* X*; if* $X \cong Y$ *then* $Y \cong X$*; and if* $X \cong Y$ *and* $Y \cong Z$ *then* $X \cong Z$*.*

Proposition 51. $\mathbb{N} \cong \mathbb{Z}$ *(as subspaces of* $\mathbb{R}$*).*

Proposition 52. $[0,1] \cong (0,1)$ *(as subspaces of* $\mathbb{R}$*).*

Proposition 53. $\mathbb{R} \cong \mathbb{P}$

Proposition 54. $\mathbb{P} \cong \mathbb{Q}$

Proposition 55. $\mathbb{Q} \cap [0,1] \cong \mathbb{Q} \cap (0,1)$

Theorem 56. *Let* X *be a set and let* d_1 *and* d_2 *be metrics for* X*. Then* d_1 *and* d_2 *are equivalent iff the identity function on* X *is a homeomorphism between* $\langle X, d_1 \rangle$ *and* $\langle X, d_2 \rangle$*.*