Proposition 27. *Let S denote the set of all non-empty bounded subsets of $\mathbb{R}^2$. For any $A \in S$ and any point $p \in \mathbb{R}^2$, let $d(p, A) \stackrel{\text{def}}{=} \inf\{\, d(p,a) : a \in A \,\}$, where the second d denotes the usual distance function in the plane. Now, for any sets $A, B \in S$, let $d(A,B)$ be the maximum of $\sup\{\, d(a,B) : a \in A \,\}$ and $\sup\{\, d(b,A) : b \in B \,\}$. Then $\langle S, d \rangle$ is a metric space.*

§ 3 *BACKGROUND ON SETS AND FUNCTIONS.*

A *set* is a collection of "objects". No further *definition* of set is really possible, since "set" is usually considered the basic undefined notion of modern mathematics. Instead of defining what a set is, one can study axiom systems that sets should satisfy, much in the same way that Euclid studied axioms for geometry as opposed to *defining* what points and angles are. For now, we simply need some terms and definitions about sets. This treatment is often referred to as "naive" set theory. We may get into some of the more difficult and interesting properties later in the semester.

The objects which make up a set are called the *elements* of the set. The notation $x \in X$ means that x is an element of the set X. If X and Y are sets, then $X \subset Y$ is read "X is a *subset* of Y", and simply means that every element of X is an element of Y. Note that every set is a subset of itself. If $X \subset Y$ and $X \neq Y$, we write $X \subsetneq Y$, and say that X is a *proper* subset of Y. The notation $X \subseteq Y$ is synonymous with $X \subset Y$, but is sometimes used to emphasize that X might in fact be equal to Y. (In olden days, some authors used $\subset$ for $\subsetneq$ in order to ease the burden on typesetters.)

Curly braces are usually used to indicate sets. The notation $\{\, 0, 1, 2, 4 \,\}$ stands for the set whose elements are exactly the integers 0, 1, 2, and 4. The notation $\{\, x : \varphi(x) \,\}$ stands for the set of all objects which satisfy the "predicate" φ. For example,

$$\{\, x : x \text{ is a real number and } x > 0 \text{ and } x < 1 \,\}$$

defines the open interval $(0,1)$. (For now, we won't worry about exactly what a "predicate" is; we'll just use common sense.) A useful abbreviation is $\{\, x \in A : \varphi(x) \,\}$ which just stands for $\{\, x : x \in A \ \ \& \ \ \varphi(x) \,\}$.

You are already familiar with certain mathematical sets: the set $\mathbb{R}$ of real numbers, the set $\mathbb{Z}$ of integers, the set $\mathbb{Q}$ of rational numbers, and also the empty set $\varnothing$ which is the unique set that has no elements in it. While we won't go into (for now) exactly what sort of objects can and can't be elements of sets, it is important to realize that sets can themselves serve as elements of other sets. For example, define some open intervals in the real line by the definition

$$I_n = (n, n+1) = \{\, x \in \mathbb{R} : n < x < n+1 \,\}$$

where n is an integer. Then we can form sets such as $\{\, I_0, I_1, I_2, I_4 \,\}$ or even $\{\, I_n : n = 0, 1, 2, \ldots \,\}$. The term *collection* is synonymous with set, but it is often used because it is less confusing to think of a "collection of sets" instead of a "set of sets". Of course, once we have collections of sets, we can form collections of collections, etc, so the distinction between set and collection only helps for "one level". As an example, you should convince yourself that $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, $\{\{\{\varnothing\}\}\}$, etc are all *different* objects.

The unordered pair containing the elements x and y is simply the set $\{\, x, y \,\}$. The *ordered pair* $\langle x, y \rangle$ is usually thought of as the elements x and y with x "first" and y "second". This can be given a precise definition as follows:

3.1 Definition.

$$\langle x, y \rangle \stackrel{\text{def}}{=} \{\, \{x\},\ \{x, y\} \,\}$$

In order to see that this definition captures the idea of ordered pair, it is only necessary to prove the following.

Theorem 28. *For any x, y, x', and y', $\langle x, y \rangle = \langle x', y' \rangle$ if, and only if, $x = x'$ and $y = y'$.*

Proposition 29. *If we throw out 3.1 and instead adopt the alternate definition $\langle x, y \rangle = \{\, x,\ \{y\} \,\}$, then Theorem 28 will still hold.*

Another common notation for ordered pair is (x, y). The advantage to using angle brackets instead of parenthesis is that we will avoid confusing pairs and intervals, i.e., $\langle 0, 1 \rangle$ is the ordered pair and $(0, 1)$ is the open interval $\{\, x \in \mathbb{R} : 0 < x < 1 \,\}$.

A *relation* is a set of ordered pairs. If R is a relation then

$$\operatorname{dom}(R) \stackrel{\text{def}}{=} \{\, x : \text{there is some y such that } \langle x, y \rangle \in R \,\}$$

$$\operatorname{ran}(R) \stackrel{\text{def}}{=} \{\, y : \text{there is some x such that } \langle x, y \rangle \in R \,\}$$

The sets $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are called the *domain* and *range* of the relation R. The relation thus tells us which elements of the domain are "related" to which elements of the range.

The largest possible relation between the sets X and Y is:

3.2 Definition.

$$X \times Y \stackrel{\text{def}}{=} \{ \langle x, y \rangle : x \in X \ \& \ y \in Y \}$$

This is called the *Cartesian product* of X and Y after René Descartes.

We usually think of a function as a rule or assignment of an element of the range to each element of the domain. This can be made more precise by defining a function to be a relation which relates exactly one element of its range to each element of its domain.

3.3 Definition. A *function* is a relation f such that for each $x \in \text{dom}(f)$ there is a unique $y \in \text{ran}(f)$ such that $\langle x, y \rangle \in f$.

This definition makes a function into a concrete object (a set) instead of an abstract rule. In order to emphasize that we *think* of functions as rules, we use the notation $f(x) = y$ as an abbreviation for the statement $\langle x, y \rangle \in f$. We also use the notation $f : X \to Y$ in order to introduce functions. This is read "f is a function from X into Y", and its meaning is: f is a function, $\text{dom}(f) = X$, and $\text{ran}(f) \subset Y$. Only requiring that $\text{ran}(f)$ be a subset of Y gives us the freedom to write such things as "define $f : \mathbb{R} \to \mathbb{R}$ by the equation $f(x) = x^2$", which really just shorthand for $f = \{ \langle x, y \rangle : x, y \in \mathbb{R} \ \& \ y = x^2 \}$ (note that $\text{ran}(f) = [0, \infty)$). If Y is equal to $\text{ran}(f)$ we say that f is a function from X *onto* Y. Note that this way of looking at a function makes it clear that a function is more than an equation. If we define $g\colon (0,1) \to \mathbb{R}$ by $g(x) = x^2$, then $g \neq f$ even though the "equations" for f and g are the same. In fact, $g \subsetneq f$! On the other hand, if we define $h\colon \mathbb{R} \to \mathbb{R}$ by $h(x) = (x+1)^2 - 2x - 1$ then (as sets) $h = f$.

A useful exercise in understanding this definition of function is to look at inverses and compositions.

3.4 Definitions. If R is a relation, then the *inverse* of R is defined by

$$R^{-1} \stackrel{\text{def}}{=} \{ \langle y, x \rangle : \langle x, y \rangle \in R \}.$$

and if R_1 and R_2 are relations, then their *composition* is defined by

$$R_2 \circ R_1 \stackrel{\text{def}}{=} \{ \langle x, z \rangle : \text{there exists some } y \text{ such that } \langle x, y \rangle \in R_1 \text{ and } \langle y, z \rangle \in R_2 \}.$$

(The order is "reversed" to make the definition agree with the standard $f \circ g(x) = f(g(x))$ equation). A function f is 1:1 means that no two members of f can have the same second element and different first elements, i.e., if $x, y \in \text{dom}(f)$ and $f(x) = f(y)$ then $x = y$.

Theorem 30. *Let f be a function. The following statements are equivalent:*

(1) *f is 1:1.*

(2) *f^{-1} is a function.*

(3) *There is a function g such that $f \circ g = \mathrm{id}_{\mathrm{dom}(f)}$ (the function $\mathrm{id}_X \stackrel{\mathrm{def}}{=} \{ \langle x, x \rangle : x \in X \}$ is the identity function on the set X).*

Proposition 31. *Let $f : X \to Y$. The following statements are equivalent:*

(1) *f is a 1:1 function from X onto Y.*

(2) *There is a function g such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.*

Finally, we review the concepts of union, intersection, and set subtraction. If A and B are sets, then

$$A \cap B \stackrel{\mathrm{def}}{=} \{ x : x \in A \ \ \& \ \ y \in B \}$$
$$A \cup B \stackrel{\mathrm{def}}{=} \{ x : x \in A \text{ or } y \in B \}$$
$$A \setminus B = A - B \stackrel{\mathrm{def}}{=} \{ x : x \in A \ \ \& \ \ y \notin B \}$$

Note that the complement of set in a metric space (1.4) is just a special case of subtraction, where the set X is "understood" to be the whole space. We have already introduced the generalizations of intersection and union to an infinite sequence of sets. This generalizes further to the concept of *arbitrary* unions and intersections.

3.5 Definition. If $\mathcal{C}$ is any collection of sets (i.e., a set each of whose elements is itself a set), then

$$\bigcap \mathcal{C} \stackrel{\mathrm{def}}{=} \{ x : x \in X \text{ for every } X \in \mathcal{C} \}$$

$$\bigcup \mathcal{C} \stackrel{\mathrm{def}}{=} \{ x : x \in X \text{ for some } X \in \mathcal{C} \}$$

It is easy to see that this generalizes the definitions in 1.3, since $\bigcap_{i=0}^{\infty} X_i = \bigcap \{ X_0, X_1, \dots \}$ and $\bigcup_{i=0}^{\infty} X_i = \bigcup \{ X_0, X_1, \dots \}$. For a trivial example which is not a sequence, let $\mathcal{C}$ denote the collection of all open intervals in $\mathbb{R}$ which contain 0. Then $\bigcap \mathcal{C} = \{0\}$ and $\bigcup \mathcal{C} = \mathbb{R}$.