Correction. Page 5: rename Theorems 10, 11, and 13 to Propositions 10, 11, and 13

§ **2** *METRIC SPACES*

2.1 Definition. A *metric space* is an ordered pair $\langle X, d\rangle$ where X is a set and d is a function which assigns a non-negative real number to each pair of points in X (i.e., $d : X \times X \to [0, \infty)$) such that for all x, y, and z in X:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y) = 0$ if, and only if, $x = y$,

(iii) $d(x, z) \le d(x, y) + d(y, z)$

Condition 3 is called the triangle inequality. The function d is called a metric or distance function for X (when it satisfies all the conditions). When the distance function is understood or unimportant will abuse notation and refer to the metric space X instead of the metric space $\langle X, d\rangle$ (i.e., the metric space $\mathbb{R}$ will mean $\langle \mathbb{R}, d\rangle$ where $d(x, y) = |x-y|$). We will refer to the elements of X as the points of the space.

Since the definitions and theorems from section 1 only referred to distance, they all generalize to the setting of metric spaces. From now on, the terms "point" and "set" will refer to points and sets in metric spaces, unless we state otherwise. To avoid unnecessary repetition, we will just add an "m" to each of the definitions and theorems from section 1 in order to refer to the corresponding more general statement. So, Definition 1.1m defines a limit point of a set in a metric space, and similarly 1.3m, 1.4m, and 1.7m define $L(A)$, $B_\varepsilon(x)$, closed, open, and bounded for metric spaces. Theorems 1m, 3m, 6m–9m, and Propositions 10m–13m are similarly the original statements generalized to metric spaces. Theorem 2 needs to be interpreted slightly differently—it says that the usual distances for $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ make those spaces into valid metric spaces.

Proposition 14. *Each of the following is a metric for* $\mathbb{R}$*:*

(a) $d(x, y) = 36.7\, |x - y|$

(b) $d(x, y) = |x + y|$

(c) $d(x, y) = 1$ for all $x, y \in \mathbb{R}$

(d) $d(x, y) = \tan^{-1}(|x - y|)$

(e) $d(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$

(f) $d(x, y) = \frac{|x-y|}{1+|x-y|}$

Proposition 15. *There is a metric for* $\mathbb{R}^2$ *such that each* $B_\varepsilon(x)$ *is "square shaped."*

Proposition 16. *(a) There is a metric for* $\mathbb{R}$ *such that* $d(x, y) = |x - y|$ *if* x *and* y *are both rational or if* x *and* y *are both irrational, but for which both* $\mathbb{Q}$ *and* $\mathbb{P}$ *are closed sets. (b) There is a metric for* $\mathbb{R}^2$ *such that* $d(x, y)$ *is the usual distance if the line connecting the points* x *and* y *passes through the origin, but such that* $x \notin L(A)$ *whenever* A *is a set which does not intersect the line through the origin which contains* x.

Proposition 17 (Howard's proposition). *If* A *and* B *are sets in a metric space, then* $L(A \cup B) = L(A) \cup L(B)$ *and* $L(A \cap B) = L(A) \cap L(B)$.

2.2 Definition. The *closure* of a set A is defined by $\mathrm{cl}(A) \stackrel{\text{def}}{=} A \cup L(A)$.

The closure of a set is the "smallest" closed set which contains A. This is made precise by:

Theorem 18. *For any set* A, $\mathrm{cl}(A)$ *is a closed set containing* A. *If* A *is a set and* B *is any closed set which contains* A, *then* $\mathrm{cl}(A) \subset B$.

2.3 Definition. Let A be a subset of a metric space X, and let $x \in X$. Then:

- x *is an interior point of* A means that there exists a positive number ε such that $B_\varepsilon(x) \subset A$.
- x *is a boundary point of* A means that for every positive number ε, $B_\varepsilon(x)$ contains a point which is in A and a point which is not in A.
- x *is an exterior point of* A means that there exists a positive number ε such that $B_\varepsilon(x)$ contains no points of A.

We will use $\mathrm{int}(A)$, $\mathrm{bd}(A)$, and $\mathrm{ext}(A)$ to denote the set of all interior, boundary, and exterior points of a set A (note that any of these might be the empty set).

Theorem 19. *Let* A *be a subset of a metric space* X, *and let* $x \in X$. *Then* x *is either an interior point of* A, *a boundary point of* A, *or an exterior point of* A. *In fact,* x *is exactly one of these.*

Note that a set is open iff every one of its points is an interior point. More than this, we have that:

Theorem 20. *The sets* $\mathrm{int}(A)$ *and* $\mathrm{ext}(A)$ *are always open, and the set* $\mathrm{bd}(A)$ *is always closed.*

Theorem 21. *For any set* A*,* $A \cup \mathrm{bd}(A) = \mathrm{cl}(A)$*.*

Proposition 22. *There is no general subset or intersection relationship between* A *and any one of the sets* $\mathrm{int}(A)$*,* $\mathrm{bd}(A)$*, and* $\mathrm{ext}(A)$*.*

Proposition 23. *There exists a non-trivial subset* A *of* $\mathbb{R}$ *such that* $\mathrm{bd}(A) = \varnothing$*. (By non-trivial, we mean* $A \neq \mathbb{R}$ *and* $A \neq \varnothing$*.)*

2.4 Definition. Let $A \subset \mathbb{R}$. A number x is an *upper bound for* A means that $x \geq a$ for all $a \in A$. A number x is a *least upper bound for* A means that x is an upper bound for A and that $x \leq y$ for every upper bound y for A. Similarly, the notions of *lower bound* and *greatest lower bound* are defined by reversing the inequalities above.

Theorem 24. *Least upper bounds and greatest lower bounds are unique (if they exist). More precisely, if* x *and* y *are both least upper bounds (or greatest lower bounds) for the set* $A \subset \mathbb{R}$*, then* $x = y$*.*

2.5 Notation. *The least upper bound of a set* A *(if it exists) is also called the supremum of* A*, and is denoted by either* $\sup(A)$ *or* $\mathrm{lub}(A)$*. Similarly, the greatest lower bound is also called the infinum and is denoted by* $\inf(A)$ *or* $\mathrm{glb}(A)$*.*

Proving the existence of $\sup(A)$ will require a careful construction of the real number system, which is a somewhat tedious task that we will put off till later. So, for now, we will accept the following "axiom" which says basically that the real number system doesn't have any "holes" in it.

2.6 Axiom. *Every set of real numbers which has an upper bound has a least upper bound.*

However, we only need this one axiom. Once we adopt it, we can then prove the corresponding statement for lower bounds.

Theorem 25. *Every set of real numbers which has a lower bound has a greatest lower bound.*

There are many examples of metric spaces which are more abstract and more difficult to "picture" than the one's we have looked at so far. Here is a start at looking at them.

Proposition 26. *Let F denote the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ (by bounded, we mean simply that the set $\operatorname{ran}(f) \stackrel{\text{def}}{=} \{\, f(x) : x \in \mathbb{R} \,\}$ has both an upper and a lower bound). Let $d(f, g) = \sup\{\, |f(x) - g(x)| : x \in \mathbb{R} \,\}$. Then $\langle F, d \rangle$ is a metric space.*