TOPOLOGY—Math 431/531
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§ **0** *INTRODUCTION*

Our class meets TR from 12:00-1:20 in room BR 420. My office is room BR 107. My "official" office hours are Monday and Friday from 1–2pm and Wednesday from 2-3pm, but I am available at other times as well. To contact me outside of class you can call my office at x3592 or send email to me at `jnorden@tntech.edu`.

This class will be taught in a style which is probably different from other mathematics courses you have had. There will be no textbook. You will receive notes containing definitions, statements, and some supplementary remarks. The statements are divided into two categories: labeled either "theorems" or "propositions." Your assignment is to provide proofs of the theorems (which are true unless I have made a typographic error) and either proofs or disproofs of the propositions (some of which are true and others of which are false).

This should not be done by looking up proofs in textbooks or papers. You will instead use the mathematical ability and knowledge you have learned in other classes and the help provided by the instructor and your fellow classmates both in and out of class. The majority of class time will be spent providing this help. You will also be asked to write up some of your proofs and distribute them to the class.

The motivation for teaching a class in this way is that you will learn to *do* mathematics instead of just learning about mathematics. The position you are placed in is similar to that of a research mathematician—you have certain facts to draw on and there are certain facts which you are trying to prove. Experience has shown that topology is a subject which is especially well suited to this method of teaching, perhaps because so many of the proofs have a "natural" flow to them. There is a long history of teaching topology in this way, starting with R. L. Moore in the 1920's. The disadvantage of this method is that we cannot cover as much material as a textbook based course can. At some point, we may address this problem by reading some book excerpts and/or research papers.

Assigning a grade can sometimes be difficult in a course like this. Ideally, everyone works hard enough to earn an A (this would certainly make my job easier!). We will have a midterm and final which together will count for about 1/4 of your final grade. But, the bulk of your grade will depend upon the work you do in trying to prove and disprove statements. Since I don't like to discuss individual grades in front of the class, it is up to you to talk with me outside of class inquire about whether your progress is satisfactory. You should feel free to do this at any time.

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§ 1 *STARTING OUT*

In order to start quickly, for this section we will make use of what you already know about the "spaces" $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$. So for this section, the terms "point" and "set" will refer only to elements or subsets of one of these three spaces.

1.1 Definition. Let A be a set and let p be a point. Then *p is a limit point of A* means that for every positive number ε there is a point a of A such that the distance from p to a is less than ε and $p \neq a$ (Some authors use the terms "cluster point" or "accumulation point" instead of limit point.)

The proof of the following theorem gives a good introduction to what proofs in topology are like.

Theorem 1. *Let A be a set. Then every limit point of the set of limit points of A is a limit point of A.*

Proof. Fix a set A, and let L denote the set of points which are limit points of A. Now fix a point x which is a limit point of L. We need to show that x is a limit point of A. Fix a positive number ε. We will be done if we can find a point of A which is not equal to x and is closer than ε to x.

Since x is a limit point of L, we can choose a point y of L such that $d(x,y) < \varepsilon$ and $y \neq x$, where $d(x,y)$ denotes the distance from x to y. Note that $d(x,y) > 0$ and $d(x,y) < \varepsilon$, so we can choose a positive number ε' such that $\varepsilon' < \min\{\, d(x,y), \varepsilon - d(x,y) \,\}$. Since y is a point of L, y is a limit point of A, so we can choose a point a of A such that $d(y,a) < \varepsilon'$. We claim that a is the point of A we are looking for. By the triangle inequality (see Theorem 2, below) we have that

$$d(x,a) \leq d(x,y) + d(y,a) < d(x,y) + \varepsilon' < d(x,y) + \varepsilon - d(x,y) = \varepsilon$$

so the distance from x to a is less than ε. Finally, $d(x,a) < \varepsilon' < d(x,y)$, so $d(x,a) \neq d(x,y)$, and thus $a \neq y$. $\square$

Note: drawing a *picture* to illustrate the arguments used above is absolutely essential for understanding the ideas which lead to the proof.

1.2 Definition. We will always use $d(x,y)$ to denote the distance between the points x and y. To be precise:

$$\text{For } \mathbb{R} := d(x,y) \stackrel{\text{def}}{=} |x-y|$$
$$\text{For } \mathbb{R}^2 = d(x,y) \stackrel{\text{def}}{=} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$
$$\text{For } \mathbb{R}^3 = d(x,y) \stackrel{\text{def}}{=} \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$$

where x_1, x_2, and x_3 denote the first, second, and (possibly) third coordinates of the point x.

We have already used the fact that $d(x,y)$ satisfies the triangle inequality, but this really does require proof.

Theorem 2. *For any points x, y, and z, $d(x,z) \leq d(x,y) + d(y,z)$.*

1.3 Notations and Definitions. The notations $A \subset B$ and $A \subseteq B$ both denote that A is a subset of B, i.e., every point which is in A is also in B (the second notation merely emphasizes the fact that every set is considered to be a subset of itself). The notations $A \cap B$ and $A \cup B$ indicate the intersection and union of A and B. More generally, if A_1, A_2, A_3, $\ldots$ is an infinite sequence of sets, then we define the infinite intersection and union of the sequence:

$$\bigcap_{n=1}^{\infty} A_i \stackrel{\text{def}}{=} \{\, x : s \in A_i \text{ for every } i = 1,2,3,\ldots\}$$
$$\bigcap_{n=1}^{\infty} A_i \stackrel{\text{def}}{=} \{\, x : s \in A_i \text{ for every } i = 1,2,3,\ldots\}$$

In general, we will use $L(A)$ to denote the set of *all* limit points of the set A (this is sometimes called the first derived set of A). Note that it may happen that A has no limit points, in which case $L(A) = \varnothing$, where $\varnothing$ denotes the "empty set" which has no elements. We let $L^2(A) = L(L(A))$, and for any integer $n > 2$, let $L^n(A) = L(L^{n-1}(A))$ (also known as the n^{th} derives set of A). As a convenience, we also let $L^1(A) = L(A)$ and $L^0(A) = A$. Finally, we let $L^\infty(A)$ denote $\bigcap_{n=1}^{\infty} L^n(A)$.

Note that Theorem 1 can now be expressed simply as $L^2(A) \subset L(A)$. This can be generalized by applying the technique of mathematical induction.

Theorem 3. *For every integer $n \geq 1$, $L^{n+1}(A) \subset L^n(A)$.*

Theorem 4. *For every integer integer $n \geq 1$ there is a set A such that $L^n(A) \neq \varnothing$ and $L^{n+1}(A) = \varnothing$*

Theorem 5. *There is a set A such that $L^n(A) \neq \varnothing$ for every positive integer n and $L^\infty(A) = \varnothing$.*

1.4 Notations and Definitions. If x is a point and $\varepsilon > 0$, the *open ball centered at x with radius ε* is denoted by $B_\varepsilon(x)$ and defined as $B_\varepsilon(x) \stackrel{\text{def}}{=} \{ p : p \text{ is a point and } d(x,p) < \varepsilon \}$. The *complement* of a set A is the set of all points which are not in A; we will denote it by A'. A set A is *closed* means that every limit point of A is also a point of A, i.e., $L(A) \subset A$. A set A is *open* means that for every point x in A there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.

Theorem 6. *A set is open iff its complement is closed. A set is closed iff its complement is open.*