Calculus III—Math 281
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Tennessee Technological University Jeffrey Norden

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§ 1 *VECTORS, POINTS, and LINEAR ALGEBRA.*

Vectors are the first topic covered this semester. At first glance, vectors seem to be very simple. It is easy enough to draw vector arrows, and the operations (vector addition, dot product, etc.) are also quite easy to learn. But as we use vectors for various purposes, they can become confusing. This is partly because the mathematical ideas underlying vectors are actually rather advanced. To give a really thorough treatment of vectors would require covering much of the material from a course on linear algebra, such as Math 451. The purpose of this first section of notes is to give a brief (but hopefully understandable) overview of the important ideas in order to help make sense out of how vectors are used in calculus.

First, what is a vector, really? A good answer, and one which points out the most important features of vectors, is that a vector is something which has "both magnitude and direction." You have probably seen this phrase before in an engineering or physics class. The problem with this answer, at least for mathematicians, is that it just leads to two other questions: "what is a magnitude?" and "what is a direction?" For a mathematics class, we need to adopt a precise definition. The standard one is:

1.1 Definition. Let n be a positive integer. An n*-dimensional vector* is an ordered list of n real numbers.

It is not hard to see how we can interpret vectors as having magnitude and direction. For example, the two-dimensional vector $\langle 2,7\rangle$ represents the idea of moving "two units to the right and then seven units up" in the Cartesian plane. We can also picture this vector as an arrow representing a displacement—start the arrow at any point you want and put its end at $(x+2,\ y+7)$ (where x and y are the coordinates of the starting point). The resulting arrow certainly has a direction. We use the length of the arrow to assign the vector a magnitude. This is easily computed via the Pythagorean theorem: $\sqrt{2^2+7^2}=\sqrt{53}$. In addition to being precise, definition 1.1 makes vector computations very easy. We add vectors "coordinate-wise", we multiply a scalar and vector by multiplying each coordinate, and we can also compute the dot and cross products of vectors without much difficulty. (The term "scalar" is just a fancy way of saying "real number." It is used to emphasize that the corresponding quantity is *not* a vector.) It is important to mention that we do not use a coordinate-wise multiplication. So we will *not* compute the product of two vectors as $\langle 2,7\rangle\langle 5,11\rangle = \langle 10,77\rangle$. You should realize that there would be nothing "illegal" or undefined about such a multiplication. It wouldn't put zeros into a denominator or cause square-roots of negative numbers to occur. The reason that coordinate-wise multiplication is not used is that it just doesn't have any useful interpretation in terms of the important vector concepts of magnitude and direction.

We are now at the point where vectors seem to be very simple and easy. What more is there to say? An important question to ask is: "What is the difference between a vector and a point?" It certainly seems that a point is very different from an arrow. We *think* of points and vectors as being different. But consider what definition 1.1 says. A two-dimensional vector, for example, is an ordered pair of numbers. But a point in the plane is also an ordered pair of numbers. So, mathematically, two-dimensional vectors and points in the plane are not different at all! The same holds for three-dimensional vectors and points in three-dimensional space. This leads to the following important observation, which may come as surprise:

1.2 Note. *Vectors and points are exactly the same mathematical objects!*

In other words, there is only one "ordered pair consisting of the number 2 followed by the number 7." This isn't changed by the fact that we now have two different notations for it: $(2,7)$ and $\langle 2,7\rangle$. How can an ordered pair simultaneously represent both a point and arrow? This sort of thing thing actually happens all the time. The number "2" might represent 2 apples, 2 centimeters, or 2 liters. We don't use a different number system for each type of thing we want to count. For a better example, consider a simple function like $f(x)=x^2$. When we write $f(\sqrt{2})=2$, the 2 on the right hand side represents the idea that "f has a magnitude of 2" at a certain x-value. When we write $f'(1)=2$, this exact same "number 2" represents the rate at which f is changing at a certain x-value. In the same way, the ordered pair "2 followed by 7" can represent either a point or the idea of moving 2 units to the right and then 7 units up.

Why is this important to understand? The fact that we can interchange vectors and points is the key to using vectors effectively. Consider the derivative example again. Even though each $f'(x)$ represents a rate, we still draw the graph of f' in order to understand its properties. We can do this because, regardless of how we interpret them, the values of $f'(x)$ are numbers. Now consider the set of all three-dimensional vectors which have a length of exactly one. By considering these as points, we see that we have described not a collection of arrows, but the surface of the unit sphere. Another way to think about how this works is that each vector is "identified" with the endpoint of the arrow drawn by starting the vector at the origin. For another example, think of the set of all vectors which are perpendicular to $\langle 1,1,1\rangle$. Convince yourself that this determines a two-dimensional flat plane in three-dimensional space. Because of this, we will use the dot-product to describe planes in three-dimensional space.

Vector notation can be confusing, and it is not particularly standardized. The textbook's usage (parenthesis for points and angle brackets for vectors) is quite popular, and I will use it in these notes and on tests. However, it is not strictly necessary to use different notations, since the context of a problem will decide whether a point or an arrow should be drawn. One alternative is to use angle brackets for both vectors and points—this eliminates the possibility of confusing a point with an open interval. More important is that we use a different notation for variables that represent vectors. This is not for distinguishing points from vectors, but rather to distinguish variables which represent scalars from those that represent vectors. When you see either $\mathbf{v} = \mathbf{u} + \mathbf{w}$ (in the textbook) or $\vec{v} = \vec{u} + \vec{w}$ (on the blackboard or my notes and tests) it *means* that in addition to whatever else is said, $\vec{u}$, $\vec{v}$, and $\vec{w}$ are vectors. This saves writing a lot of tedious phrases like "let u, v, and w be vectors." The length of a vector is sometimes indicated as $|\vec{v}|$ (our textbook does this). You then have to use the context to distinguish vector magnitude from a simple absolute value. I will use the double bars instead: $\|\vec{v}\|$ is just an alternate notation for the same thing ($\| \langle 2,7\rangle \| = | \langle 2,7\rangle | = \sqrt{53}$), but it emphasizes that we are using vectors. Often, $\|\vec{v}\|$ is read as "the *norm* of $\vec{v}$", since the term "norm" is commonly used in linear algebra to denote the magnitude of a vector. Finally, recall that $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^2$ denotes the set of all points in the Cartesian plane, and $\mathbb{R}^3$ denotes the set of all points in three-dimensional space. We also use $\mathbb{R}^2$ and $\mathbb{R}^3$ to denote the set of all two-dimensional and three-dimensional vectors respectively. To emphasize once more: $\mathbb{R}^2$ *really* denotes the set of all possible ordered pairs of real numbers, and $\mathbb{R}^3$ is really just the set of all possible ordered triples of numbers. It is also now clear how we can define higher-dimensional spaces $\mathbb{R}^4$, $\mathbb{R}^5$, etc., even though nobody can really visualize what such spaces would look like.

To illustrate these notations, consider the set defined by:

$$S = \{\, \vec{v} \in \mathbb{R}^3 : \|\vec{v}\| = 1 \,\}$$

Recall that "$\in$" is used to denote the "is an element of" relation. The above equation is usually read as: "S equals the set of all $\vec{v}$ in $\mathbb{R}^3$ such that $\|\vec{v}\| = 1$." This is the same example we considered above—the unit sphere. Note that it is more precise than just writing the equation $\|\vec{v}\| = 1$. With vector equations, you often have to use the context to decide which dimension to use. The same equation $\|\vec{v}\| = 1$ might also mean $\{\, \vec{v} \in \mathbb{R}^2 : \|\vec{v}\| = 1 \,\}$, which is the unit circle in the plane. As another example, $P = \{\, \vec{v} \in \mathbb{R}^3 : \vec{v} \cdot \langle 1,1,1\rangle = 0 \,\}$ defines the plane that we mentioned before. As a final example, convince yourself that $L = \{\, \vec{v} \in \mathbb{R}^3 : \vec{v} = t\,\langle 1,2,3\rangle \text{ for some } t \in \mathbb{R} \,\}$ defines a line in three-dimensional space.

The field of mathematics called *linear algebra* studies the general notion of a *vector space.* While this topic isn't really necessary for the calculus we will study this semester, it may help to at least know about the definitions. The "dangerous bend" symbol at the start of this paragraph is to let you know that I'm discussing a more advanced topic. Please don't think of this as meaning that the information is optional—rather keep in mind that it's ok if you don't understand all the details on first reading. You can also comfort yourself with the fact that such material will only occur on extra-credit problems. (I should mention that this clever symbol is not my own invention. It was developed by Donald Knuth who, among other things, invented the $\TeX$ typesetting system that I use for these notes.)

In linear algebra, a *vector space* is defined to be a set V together with two operations defined on it: scalar multiplication and vector addition. So for any number t and any $\vec{v}$ in V there must be a well defined $t\vec{v}$ in V, and for any $\vec{v}$ and $\vec{w}$ in V, there must be a well defined $\vec{v}+\vec{w}$ in V. The set V and the multiplication and addition defined on it can be quite general and abstract. But, in order to earn the title of "vector space" they are required to satisfy some algebraic properties, similar to the commutative and distributive properties that the real numbers have. These properties are exactly the ones which are listed in the box at the bottom of page 692 in our textbook. Any set with appropriate operations is called a vector space. The restriction of definition 1.1 no longer holds in linear algebra.

For an example of a general vector space, consider letting V denote the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Define tf and $f+g$ by $(tf)(x) \stackrel{\text{def}}{=} tf(x)$ and $(f+g)(x) \stackrel{\text{def}}{=} f(x)+g(x)$. This does satisfy the required algebraic properties (check this), but it is certainly not a set of ordered pairs, triples, etc. This space is not 2-dimensional, or even a 200-dimensional. In an important sense, this is an *infinite* dimensional space.

We won't go very far into the general theory of linear algebra this semester, but you are likely to encounter it in later classes. For now, the important thing is that it provides the best answer to the question "What is a vector?" A vector is an element of a vector space. So, when we think of $\mathbb{R}^2$ as the plane with its geometric properties (such as distance) we call it "Euclidean space" and refer to its elements as "points." But if we think of $\mathbb{R}^2$ as a set on which scalar multiplication and vector addition are defined, we call it a "vector space" and refer to its elements as "vectors."

1.3 Exercises. Decide which of the following are well defined, and evaluate those that are. Give a geometric interpretation for each of the defined answers.

1) $2\langle 1,-2,\pi\rangle - \sqrt{2}\,\langle 8,4,-7\rangle$ 2) $\langle 1,3\rangle + \langle 5,2,1\rangle$ 3) $\langle 1,4\rangle\,\langle 6,2\rangle$ 4) $\langle 1,4\rangle \cdot \langle 6,2\rangle$
5) $\langle 1,4\rangle \cdot \langle 6,2,3\rangle$ 6) $\langle 1,4\rangle \times \langle 6,2\rangle$ 7) $\langle 1,4,1\rangle \times \langle 6,2,3\rangle$ 8) $\langle 1,4,1\rangle \cdot \vec{\imath}$
9) $(2\vec{\imath}+3\vec{\jmath})\cdot(7\vec{\imath}-8\vec{\jmath})$ 10) $(2\vec{\imath}+3\vec{\jmath})\times(\vec{\imath}-\vec{\jmath}+2\vec{k})$ 11) $\langle 1,2,3,4,5,6,7\rangle \cdot \langle 7,6,5,4,3,2,1\rangle$
12) $\|\langle 1,2,3,4,5,6,7\rangle\|$ 13) $\left\|\left\langle \sqrt{2},10,1/\pi\right\rangle\right\|$ 14) $\vec{r}\cdot\vec{r} - \|\vec{r}\|^2$